Notes on Q1-5 of the Nov. 2014 MAT Paper

(13 Pages; 28/10/16)

(to be read in conjunction with the official solutions)

Q1/A

As a slight variation on the official solution, we can say:

$y^2 - 8y - 9 < 0 \Rightarrow (y-9)(y+1) < 0 \Rightarrow -1 < y < 9$ (eg from the graph), where $y = x^2$

$\Rightarrow x^2 < 9$

Q1/B

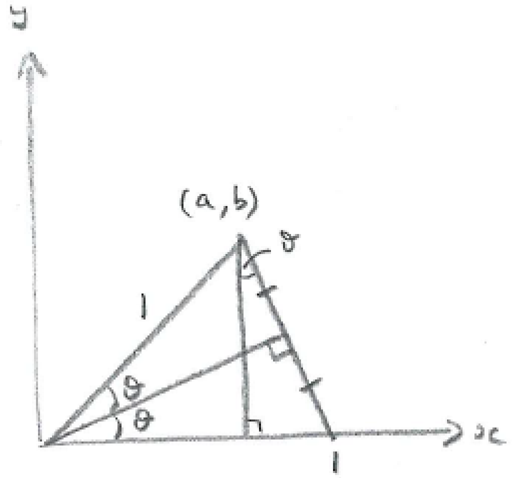
A standard tactic for tackling problems where the required approach is not obvious is to look for something (reasonably worthwhile) that can be done quickly. In this case, we can complete the square. As per the official solution, this provides evidence to eliminate all but one of the possible answers.

Q1/C

The official solution says that the 2nd derivative needs to be positive, in order to establish the minimum. For a general function, this isn't true. In the case of $y = x^4$, for example, the 2nd derivative is zero at the minimum. However, all cubics have a point of inflexion, when the 2nd derivative is zero, and so we only need to consider positive 2nd derivatives.

Q1/D

Alternative Method 1



From the diagram above, the x-coordinate of the reflected point is

$1(\cos(2\theta)) = cos^2\theta - sin^2\theta = 2cos^2\theta - 1 = \frac{2}{sec^2\theta} - 1$

$= \frac{2}{1+tan^2\theta} - 1 = \frac{2}{1+m^2} - 1 = \frac{2-(1+m^2)}{1+m^2} = \frac{1-m^2}{1+m^2}$,

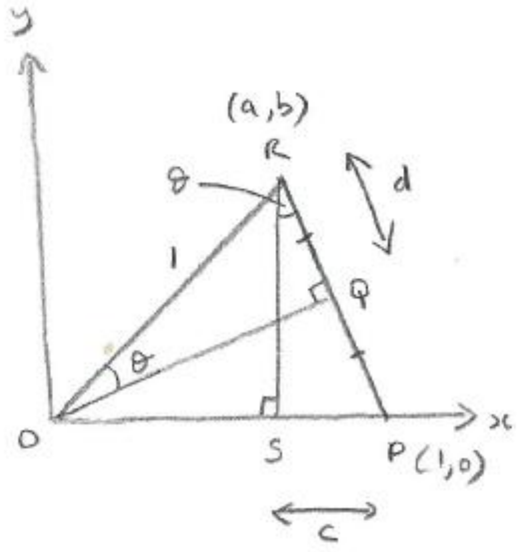
which shows that (d) is the correct answer.

Also, y-coordinate is $1(\sin(2\theta)) = 2sin\theta cos\theta$

$tan\theta = \frac{m}{1} \Rightarrow sin\theta = \frac{m}{\sqrt{1+m^2}}$ and $cos\theta = \frac{1}{\sqrt{1+m^2}}$ (by Pythagoras)

so that $2sin\theta cos\theta = \frac{2m}{1+m^2}$, as required.

Alternative Method 2



From the diagram above, triangle OQR gives $sin\theta = \frac{d}{1}$ and triangle PRS gives $\frac{b}{2d} = cos\theta$

Then $b = 2sin\theta cos\theta$ (1)

Also $c^2 = (2d)^2 - b^2 = (2sin\theta)^2 - 4sin^2\theta cos^2\theta$

$= 4sin^2\theta(1 - cos^2\theta) = 4sin^4\theta$

so that $c = 2sin^2\theta$

and $a = 1 - c = 1 - 2sin^2\theta = 1 - 2(1 - cos^2\theta) = 2cos^2\theta - 1$ (2)

Then from (1) & (2) we can proceed as in the previous method.

Q1/E

The completing of the square can be carried out as follows:

$5 + 4cosx - 4cos^2x = -4(cos^2x - cosx) + 5$

$= -4\left(cosx - \frac{1}{2}\right)^2 + 1 + 5 = 6 - (2cosx - 1)^2$

Q1/F

This problem can be investigated either algebraically, or graphically. In fact a combination of the two approaches is probably the quickest way of arriving at the answer.

From an algebraic point of view, any combination of the functions S and T will produce a function of the form $a \pm x$ (where a is an integer), so that t has to be odd, in order to give the $-x$ in

$F(x) = 8 - x$.

From a graphical point of view, if x represents the initial position on the x-axis, with T having the effect of a reflection in the line $x = 0$, and S that of a translation of one place to the right, then $F(x)$ can be considered to represent the final position.

$8 - x$ can be written as $-(x - 8)$, so that if x is odd/even, $8 - x$ will be odd/even as well; ie there is an even difference between the final and initial positions in both cases, and so s must be even, as any reflections about $x = 0$ have no effect on the odd/even status.

Thus t is odd and s is even, so that the answer is (c).

Q1/G

$([1 + xy] + y^2)^n = (1 + xy)^n + n(1 + xy)^{n-1}y^2$

$+\binom{n}{2}(1 + xy)^{n-2}y^4 + \cdots$

Only the term $n(1 + xy)^{n-1}y^2$ will contain a power of y that is 2 greater than the power of x, and we require the term $n\binom{n-1}{3}(xy)^3y^2$ within this, so that the required coefficient is

$n\binom{n-1}{3} = \frac{n(n-1)!}{3!(n-4)!}$, which can be seen to equal (d):

$$\frac{n(n-1)!}{3!(n-4)!} = \frac{n!(4!)}{3!(n-4)!(4!)} = 4\binom{n}{4}$$

Alternatively, knowledge of the trinomial expansion:

$$(a+b+c)^n = \sum_{\substack{i,j,k \\ (i+j+k=n)}} \binom{n}{i,j,k} a^i b^j c^k,$$

where $\binom{n}{i,j,k} = \frac{n!}{i!j!k!}$, gives the answer straightaway,

as we require the term $\binom{n}{n-4,3,1}(1)^{n-4}(xy)^3(y^2)^1$,

so that the coefficient is

$$\binom{n}{n-4,3,1} = \frac{n!}{(n-4)!(3!)(1!)} = \frac{n!(4!)}{(n-4)!(3!)(4!)} = 4\binom{n}{4}$$

Q1/H

For questions of this type, it is often necessary to calculate a few values of $F(n)$, in order to get a feel for the problem. However, in this case, it is just a matter of seeing how much has been added on to $F(1)$ by the time we reach $F(6000)$. As per the official solution, the numbers from 2 to 6000 can be categorised as "even", "multiples of 3" and "multiples of 6", and hence split up into the distinct sets in the questions ("2 divides n but 3 does not divide n" etc).

Q1/I

At each stage of a composite transformation, we must always be doing one of the following:

(a) replacing x with $x + a$ (where a can be negative), to give a translation of $\binom{-a}{0}$

(b) replacing x with kx (where k can be negative; eg $k = -1$ represents a reflection in the y-axis), to give a stretch of factor $1/k$ in the x-direction (ie the graph is seen to compress if $k > 1$)

(c) replacing y with $y + a$ (or, equivalently, subtracting a from the function, so that $y = f(x) \rightarrow y = f(x) - a$),), to give a translation of $\binom{0}{-a}$

(d) replacing y with ky (or , equivalently, dividing the function by k, so that $y = f(x) \rightarrow y = \frac{1}{k}f(x)$), to give a stretch of factor $1/k$ in the y-direction

Q1/J

[In the official solution, $\int_{-1}^{1} f(x)du$ should read $\int_{-1}^{1} f(x)dx$ (or $\int_{-1}^{1} f(u)du$) etc]

$\int_{-1}^{1} f(t)dt$ and $\int_{-1}^{1} f(x)dx$ can both be written as a constant, A say.

A possible line of investigation could be substituting in particular values of x (eg $x = -1$ & 1), or (more generally) $x = a$:

This yields $6 + f(a) = 2f(-a) + 3a^2A$ and

$6 + f(-a) = 2f(a) + 3a^2A$

Subtracting one from the other then gives $f(a) = f(-a)$.

As this is true for all a, we can say that $f(x) = f(-x)$.

Then, integrating both sides of the original equation from -1 to 1:

$\int_{-1}^{1} 6\, dx + \int_{-1}^{1} f(x) dx = 2\int_{-1}^{1} f(x) dx + 3A\int_{-1}^{1} x^2 dx$

$\Rightarrow [6x]_{-1}^{1} + A = 2A + 3A[\frac{1}{3}x^3]_{-1}^{1}$

$\Rightarrow \left(6 - (-6)\right) = A + A(1 - (-1))$

$\Rightarrow 12 = 3A$

$\Rightarrow A = 4$

[The official solution points out that $\int_{-1}^{1} f(-x) dx = \int_{-1}^{1} f(x) dx$, as an alternative method.]

Q2

For (i): Alternatively, $x = 1 \Rightarrow b^2 - 2b - 1 = -a^2 < 0$, and we then require b to lie between the two roots of $b^2 - 2b - 1 = 0$,

For (iii): We can write $x^3 + 2bx^2 - a^2x - b^2 = (x - c)^2(x - 1)$. Then equating coefficients of x^0 & x^2 leads to $b = c = -\frac{1}{4}$

Strictly speaking we need to check that a solution exists for a, and we can equate coefficients of x to do this.

Q3

For (iv):

It isn't always clear whether we need to be looking for additional equations or inequalities that can be deduced from the situation. Generally speaking, always assume the simplest interpretation:

so, unless there is anything very obvious, aim to work with the result(s) that have already been established. (Were there to be anything else, it is likely that this would have been proved in another part of the question, or its use hinted at.)

We know that $\frac{1}{2n}\left(\frac{b+1}{b-1}\right)(a-1) \geq I$, and that $I = a - 1$,

with $b^n = a$

The simplest thing to do is to write $\frac{1}{2n}\left(\frac{b+1}{b-1}\right)(a-1) \geq a - 1$

and then $\frac{1}{2n}\left(\frac{b+1}{b-1}\right) \geq 1$ $[A]$,

since $a = I + 1 > 1$ (as $f(x) > 0$ and hence $I = \int_0^1 f(x)dx > 0$)

[We can't be sure how particular the examiners are going to be about making and justifying statements such as $a - 1 > 0$; it could perhaps be something to leave to the end, given the time pressure.]

The required result to prove [the abbreviation rtp could be used; ideally having defined it the first time] can be rewritten as

$b \leq 1 + \frac{2}{2n-1} = \frac{(2n-1)+2}{2n-1} = \frac{2n+1}{2n-1}$ (B)

It then looks promising to make b the subject of [A]:

$\frac{1}{2n}\left(\frac{b+1}{b-1}\right) \geq 1 \Rightarrow b + 1 \geq 2n(b-1)$

$\Rightarrow b(1-2n) \geq -2n - 1$

$\Rightarrow b \leq \frac{-2n-1}{1-2n}$ (as $1 - 2n < 0$)

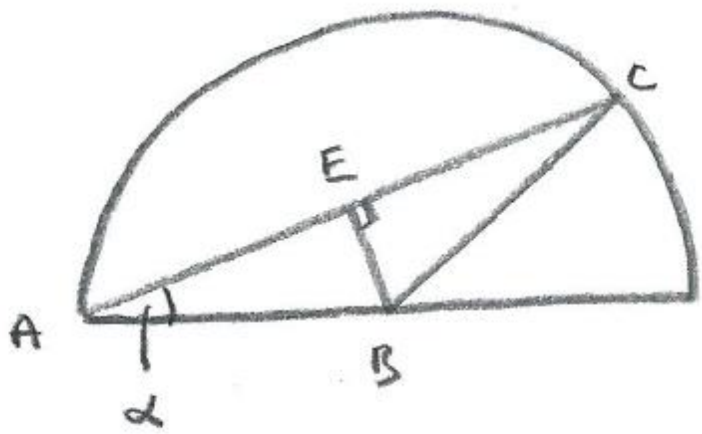
$\Rightarrow b \leq \frac{2n+1}{2n-1}$, which is (B).

Q4

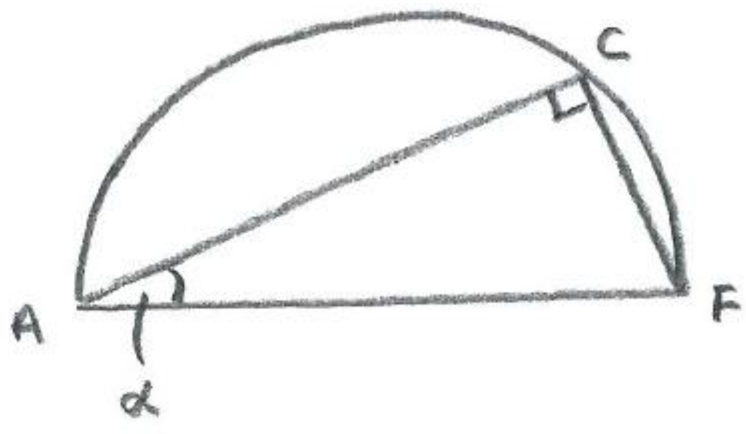
For (i), there are some alternative methods (in addition to the one in the official solution):

(a) Dropping a perpendicular from B to AC divides ABC into two equal right-angled triangles (see diagram below). If the foot of the perpendicular is at E, $AE = ABcos\alpha = cos\alpha$, whilst $EB = ABsin\alpha = sin\alpha$. Then the area of AEB is $\frac{1}{2}AE.EB = \frac{1}{2}cos\alpha\ sin\alpha$, so that the area of ABC is $cos\alpha\ sin\alpha$

$= \frac{1}{2}sin(2\alpha)$



(b) A right-angled triangle ACF can be created, as in the diagram below.



Then $AC = AFcos\alpha = 2cos\alpha$, and the area of ABC is

$\frac{1}{2}AB.AC\ sin\alpha = \frac{1}{2}(1)(2cos\alpha)sin\alpha = \frac{1}{2}\sin(2\alpha)$

For (iv), we are effectively being asked to show that area of ABX is $\frac{\sin(2\beta)sin\alpha}{2\sin(2\beta-\alpha)}$ (A)

The approach: area of ABX is $\frac{1}{2}AB.AX.sin\alpha = \frac{1}{2}AXsin\alpha$ looks promising, because of the common term of $sin\alpha$. Thus we just need to show that

$AX = \frac{\sin(2\beta)}{\sin(2\beta-\alpha)}$, which suggests the use of the sine rule.

At this point we could note that we know another side of the triangle AXB (namely $AB = 1$), whereas we don't know another side of the triangle AXD.

Then $\angle ABD = \pi - 2\beta$ (since triangle ABD is isosceles) and

$\angle AXB = \pi - (\alpha + \angle ABD) = \pi - \alpha - (\pi - 2\beta) = 2\beta - \alpha$

Then the sine rule gives $\frac{AX}{\sin(\pi-2\beta)} = \frac{1}{\sin(2\beta-\alpha)}$

so that $AX = \frac{\sin(\pi-2\beta)}{\sin(2\beta-\alpha)} = \frac{\sin(2\beta)}{\sin(2\beta-\alpha)}$, as required.

Part (v) is a bit vague, in that we are being invited to write down some expression for the area of ABX, in order to derive an unspecified formula for F. Also, it isn't immediately clear whether the F mentioned is the same as the F in parts (ii) - (iv), or whether it is the corresponding expression in the new situation where $0 < \beta < \alpha$ [the latter must be the correct interpretation, based on how F is defined in (ii)].

ABX was originally found to be $\frac{\sin(2\beta)sin\alpha}{2\sin(2\beta-\alpha)}$ [result (A) above]

In the new configuration, with $0 < \beta < \alpha$, we need to swap the roles of α and β, by the definition of X, to give $\frac{\sin(2\alpha)sin\beta}{2\sin(2\alpha-\beta)}$

[The question asks us to write down this expression, but it isn't obvious - except in retrospect - that we are expected to consult the working to part (iv) in order to find the area of ABX. Conceivably we could have been expected to extract the area of ABX from the definition of F in (ii).]

The formula for ABC is unchanged, being $\frac{1}{2}\sin(2\alpha)$, and so,

from the definition of F in (ii), $F = \frac{Area\ of\ ABX}{Area\ of\ ABC} =$

$\frac{\sin(2\alpha) sin\beta}{2\sin(2\alpha-\beta)}\left(\frac{2}{\sin(2\alpha)}\right)$

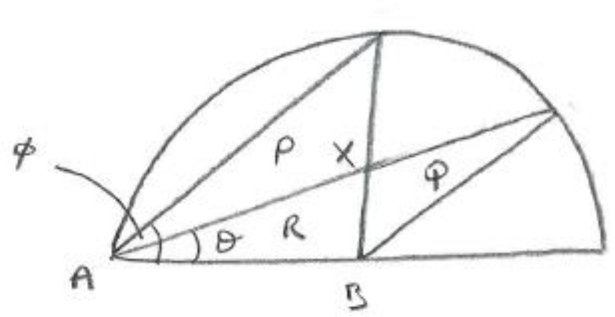
$= \frac{sin\beta}{\sin(2\alpha-\beta)}$

Alternative Method for (v)

A possibly simpler approach is the following (but it could only be used as a check, as we aren't following the question-setter's preferred method - even though they are probably just trying to be helpful):

In the diagram below, we are being asked to find $\frac{R}{R+P}$

where $\theta = \beta\ \ and\ \phi = \alpha$



We know, from (iv) that $\frac{R}{R+Q} = \frac{\sin(2\phi) sin\theta}{\sin(2\phi-\theta)\sin(2\theta)}$

Also $R + Q = \frac{1}{2}\sin(2\theta)$ and $R + P = \frac{1}{2}\sin(2\phi)$

Then $\frac{R}{R+P} = \frac{R}{R+Q}\left(\frac{R+Q}{R+P}\right) = \frac{\sin(2\alpha) sin\beta}{\sin(2\alpha-\beta)\sin(2\beta)}\left(\frac{\frac{1}{2}\sin(2\beta)}{\frac{1}{2}\sin(2\alpha)}\right) = \frac{sin\beta}{\sin(2\alpha-\beta)}$

Q5

(i) AAA, AAB, ABA, ABB, ABC

(ii) $c_{n,1} = 1$: The letters are all A.

$c_{n,n} = 1$: The n letters have to be in alphabetical order.

(iii) Case 1: The final position is different from the others.

There are $c_{n-1,k-1}$ possibilities for the other positions, and just one for the final position (as it has to be a new letter);

ie $c_{n-1,k-1}(1)$ possibilities

Case 2: The final position is the same as one of the others.

There are $c_{n-1,k}$ possibilities for the other positions, and k

possibilities for the final position;

ie $c_{n-1,k}(k)$ possibilities;

giving a total of $c_{n-1,k-1}(1) + c_{n-1,k}(k) = kc_{n-1,k} + c_{n-1,k-1}$

(iv) $r_{\text{n}} = \sum_{k=1}^{\text{n}} c_{\text{n},k}$

$r_4 = \sum_{k=1}^{4} c_{4,k} = c_{4,1} + c_{4,2} + c_{4,3} + c_{4,4}$

$= 1 + (2c_{3,2} + c_{3,1}) + (3c_{3,3} + c_{3,2}) + 1$

$= 2 + c_{3,1} + 3c_{3,2} + 3c_{3,3}$

$= 2 + 1 + 3(2c_{2,2} + c_{2,1}) + 3$

$= 6 + 3(2 + 1)$

$= 15$

(v) [The official sol'ns gives a quick 'lateral thinking' way of answering this question; not using (iii). The form of words "Give a formula ... Justify your answer" perhaps suggests that (iii) needn't be used.]

From (iii), $c_{n,2} = 2c_{n-1,2} + c_{n-1,1}$

$= 2(2c_{n-2,2} + c_{n-2,1}) + 1 \quad$ for $n > 2$

$= 4c_{n-2,2} + 2 + 1$

$= 4(2c_{n-3,2} + c_{n-3,1}) + 2 + 1 \quad$ (for $n > 3$)

$= 8c_{n-3,2} + 4 + 2 + 1$

$\ldots = 2^{n-2}c_{n-(n-2),2} + 2^{n-3} + \cdots + 2 + 1 \quad$ for $n > 2$

$= 2^{n-2} + \cdots + 2 + 1$

$= \frac{2^{n-1}-1}{2-1}$

$= 2^{n-1} - 1$

And for $n = 2$, $2^{n-1} - 1 = 1 = c_{2,2}$

Thus $c_{\mathrm{n},2} = 2^{n-1} - 1$ for all $n \geq 2$