Notes & Solutions for Q1-5 of the Nov. 2013 MAT Paper

(13 pages; 27/10/16)

(to be read in conjunction with the official solutions)

Q1/B

A reflection in the line $x = L$ is equivalent to (A) a reflection in the line $x = 0$, followed by (B) a translation of $\begin{pmatrix} 2L \\ 0 \end{pmatrix}$ [after (A), the two graphs meet when $x = 0$; we want the image of $(0, f(0))$ [where $f(x)$ is the original function] to be at $(2L, f(0))$, so a translation of 2L is necessary]. Similarly for a reflection in the line $y = M$.

So , for a reflection in $x = \pi$, (A) gives $y = \sin(-x) = -sinx$; then (B) gives $y = -\sin(x - 2\pi) = -sinx$

Then, for a reflection in $y = 2$, the equivalent of (A) gives $-y = -sinx$, or $y = sinx$, and the equivalent of (B) then gives

$y - 4 = sinx$, or $y = 4 + sinx$

Q1/C

This question really requires knowledge of the Chain Rule, for a full understanding.

As an example of a general choice of strategies, we could start with the information that we are given; ie $f'(x) = g(x + 1)$ etc, and aim to find $f''(2x)$ (this is what the official solution does), or we could start with $f''(2x)$ and try to use the information given.

The important point to note is that $f'(2x)$ means $\frac{d}{du}f(u)$, where $u = 2x$; so that the differentiation is with respect to $2x$ (rather than x).

Alternative Solution

$f''(2x) = f''(u)$, where $u = 2x$

and $f''(u) = \frac{d}{du}f'(u) = \frac{d}{du}g(w)$, where $w = u + 1$

Then $\frac{d}{du}g(w) = \frac{d}{dw}g(w).\frac{dw}{du} = \frac{d}{dw}g(w) = g'(w) = h(w-1)$

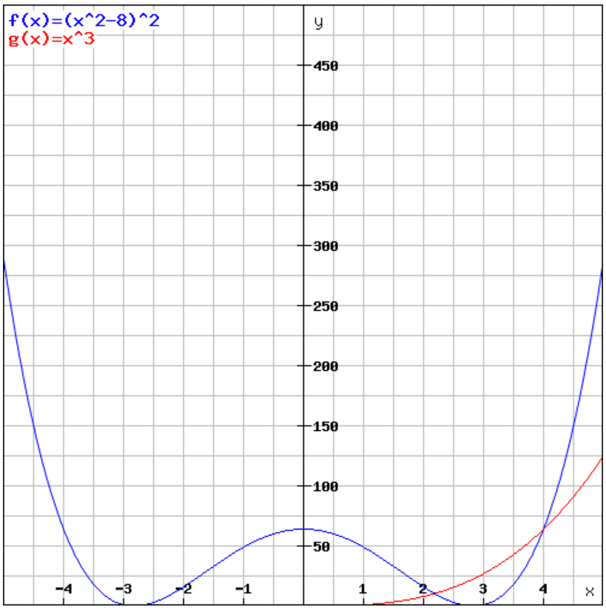
So $f''(2x) = h(w-1) = h(u) = h(2x)$

Q1/D

(a), (c) & (d) can just be eliminated by considering $y = 0$ and $x = 0$

Q1/F

As per the official solution, the logarithm equations can be converted to equations involving powers, which are likely to be more useful. The most natural thing to do at this point is to derive an equation for a (since the answers are concerned with the number of solutions for a). This leads to $a^3 = (a^2 - 8)^2$. We only need to establish the number of solutions, and we could (in theory - though probably not within the time constraints of the exam) sketch the graphs of $y = x^3$ and $y = (x^2 - 8)^2$ (see below). This reveals two points of intersection, making (b) a tempting answer. However, it turns out that the smaller value of a produces a negative value for b. If this is spotted, then we arrive at the correct answer of (a).



With hindsight, this approach should have been abandoned just prior to any sketching, on the grounds that it is too time-consuming. The official solution shows that if instead we obtain an equation for b: $b^3(b-1) = 8$, then we can avoid any sketching by noting that $y = x^3(x-1)$ is an increasing function, which is negative for $0 < x < 1$ and > 8 for large x. Hence there is just one solution for b. We still have to check, however, that we have positive values for a and c. At this point the solution becomes less satisfactory, as it relies on looking for a convenient (integer) solution for b. Bearing in mind how the question is likely to have been created, this is still a worthwhile method though (and in this case, b turns out to equal 2).

Note for future reference, that the product of two positive increasing functions, such as $y = x^3$ and $y = x - 1$ is itself an increasing function.

Q1/H

$y = \sqrt{2 - x^2}$ is the equation of a circle; the other graph is that of a straight line.

Having drawn a diagram (as in the official solution), only one of the multiple choice options has a form consistent with the required area of a sector minus the area of a triangle.

If the solution is completed, it is noted that the coordinates of the intersection of the circle and line are very convenient (being (1,1)). Questions are often designed to be easily solvable in this way - though the simplification is not apparent initially.

Q1/I

The quickest way of making progress with this type of question is simply to start writing out the terms.

In this case, this establishes that each of the terms is either 1 or -1 (though this could have been deduced anyway).

Unfortunately the pattern of 1 and -1s does not become clear until a large number of terms has been written out. If you happen to go down this route (as the official sol'n does) then, once the pattern has been established, it is a simple matter to find the overall sum.

An alternative approach that doesn't rely on a pattern emerging is as follows:

$\sum_1^{100} F(k) = 1 + 2[F(2) + F(3) + \cdots + F(49)] + F(50)$

Then F(2)+F(3)=0 and F(50)=F(25)=F(12)=F(6)=F(3)= -1

so that $\sum_1^{100} F(k) = 2[F(4) + \cdots F(49)]$

$= 4[F(2) + \cdots F(24)] = 4[F(4) + \cdots F(23)] + 4F(24)$

$= 8[F(2) + \cdots + F(11)] + 4F(3)$

$= 8[F(4) + \cdots F(11)] - 4$

$= 16[F(2) + \cdots F(5)] - 4$

$= 16(F(4) + F(5)] - 4$

$= 16(1 + 1) - 4$

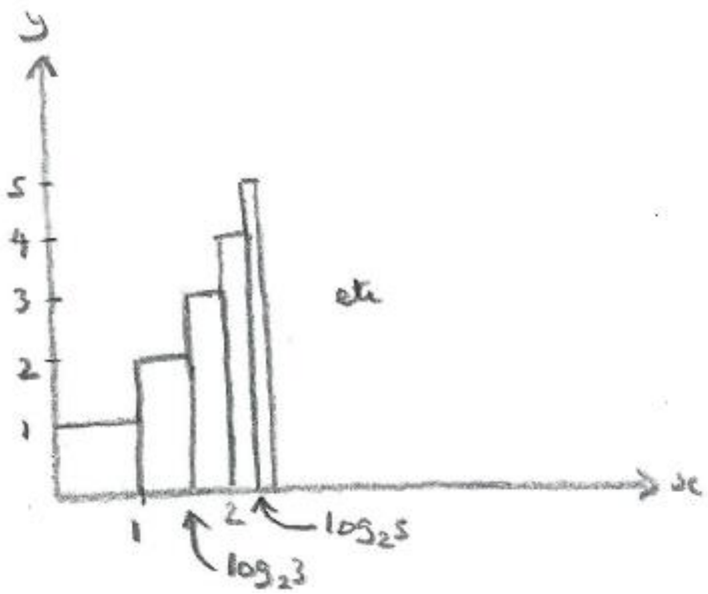
$= 28$

Q1/J

[Answer = (b); not (d) as stated in the official solution (though the solution itself is correct).]

The 'floor' function $[x]$ is sometimes written as $\lfloor x \rfloor$, with the 'ceiling' function (the smallest integer greater than or equal to x), being written as $\lceil x \rceil$ (though these latter symbols do look like printing errors).

The advantage of setting n equal to 3 (say) is that the process is often quicker than for a general n, and so less time is wasted if nothing comes of it.



Summing the rectangles represented by the integral gives:

$\int_0^3 [2^x]dx = 1(log_2 2) + 2(log_2 3 - log_2 2) + 3(log_2 4 - log_2 3) +$...

$+7(log_2 8 - log_2 7)\ )$

$= 7log_2 8 - log_2 7 - log_2 6 - \cdots - log_2 2,$ so that

$\int_0^n [2^x]dx = (2^n - 1)log_2(2^n) - log_2(2^n - 1) - log_2(2^n - 2) - \cdots - log_2 2$

$= (2^n - 1)n - log_2((2^n - 1)!)$

[Referring to the multiple choice options:]

$= n2^n - log_2 2^n - log_2((2^n - 1)!)$

$= n2^n - log_2((2^n)!)$

Q2

(i) $f(t) - kf(1-t) = t$ (1)

Replacing t with $1 - t$ gives $f(1-t) - kf(t) = 1 - t$ (2)

Subst. from (2) into (1): $f(t) - k\{1 - t + kf(t)\} = t$

$\Rightarrow f(t)\{1 - k^2\} = t + k(1-t)$ (3)

$\Rightarrow f(t) = \frac{t(1-k)+k}{1-k^2}$ (as $k^2 \neq 1$) [the eq'n of a straight line]

(ii)(a) If $f(t) - f(1-t) = t$, then from (3) in (i), with $k = 1$,

$0 = t + (1-t) = 1$

Hence $f(t) - f(1-t) = t$ is not possible.

(b) Suppose that $f(t) - f(1-t) = g(t)$ (4)

Then, replacing t with $1-t$ gives

$f(1-t)-f(t)=g(1-t)$ (5)

Adding (4) & (5): $0=g(t)+g(1-t)$

Thus, $g(t)$ must satisfy the condition $g(1-t)=-g(t)$ (A)

[Note that - in theory - there might be other conditions that $g(t)$ must satisfy. We have only shown that (A) is a necessary condition; not that it is a sufficient condition for (*). The question is slightly ambiguous: is 'condition' intended to relate to a single condition (one of possibly many). Or - if there were multiple conditions - does 'condition' mean 'the set of requirements'?

As a simpler example, if we were asked to give the condition that a number had to satisfy, in order for it to be a multiple of 9, we could say that the digits had to add up to a multiple of 3; but that wouldn't be in the spirit of the question.

At the end of the day, MAT questions are not concerned with the finer points of semantics.

For this question, it doesn't seem to be that straightforward to show that (A)$\Rightarrow$ $(*)$; for example, by finding an expression for $f(t)$ in terms of $g(t)$, for which (*) held.]

(c) [The approaches mentioned in the official sol'n don't seem to be that obvious.]

If we suppose that $f(t)$ is a polynomial in t (bearing in mind that it is a straight line in (i)), then when we equate coefficients for (*), we will have a cubic (or lower order polynomial).

Suppose that $f(t)=At^3+Bt^2+Ct+D$,

so that $f(t)-f(1-t)=(At^3+Bt^2+Ct+D)$

$-[A(1-t)^3 + B(1-t)^2 + C(1-t) + D] = (2t-1)^3$ (B)

[At this stage, it is worth seeing if any useful observations can be made. Perhaps there is a simplified solution, with some of the coefficients equal to zero.]

As D cancels, it can take any value; such as 0.

[Expanding out everything and equating coefficients will be reliable, but quite time-consuming.]

Equating coefficients of t^3 in (B): $A + A = 8$, so that $A = 4$

Let $t = 1$ in (B), to give $A + B + C = 1$, and hence $B + C = -3$

[$t = 0$ gives the same result, unfortunately; as does $t = 2$]

Equating coefficients of t in (B): $C + 3A + 2B + C = 6$; once again giving the same result.

So it would appear that any cubic where $B + C = -3$ may well be a solution. Equating coefficients of t^2 will confirm this:

$B - 3A - B = -12$, which is consistent

So let $B = 0$ & $C = -3$, say, to give $f(t) = 4t^3 - 3t$

[and we can check that this satisfies (B)]

Q3

Solution for (iv):

From (iii), $f_{2-k}(x) = -f_k(1+t)$, where $x = 1 - t$, so that

$t = 1 - x$ and $1 + t = 2 - x$

Thus $f_{2-k}(x) = -f_k(2-x)$

Replacing x with $2 - x$ gives a reflection in $x = 1$, and so the required transformation is a reflection in $x = 1$, together with a reflection in the x-axis, which is also equivalent to a rotation of 180° about (1,0); again as discussed earlier.

Figure 1 below shows the reflection in $x = 1$ (for $k = 0.75$). Figure 2 shows the reflection in the x-axis, making it clear that the total area either side of the x-axis hasn't changed.

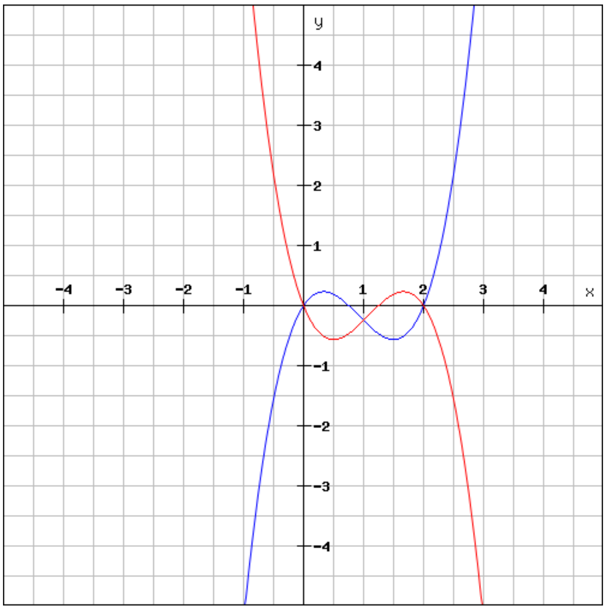


Figure 1

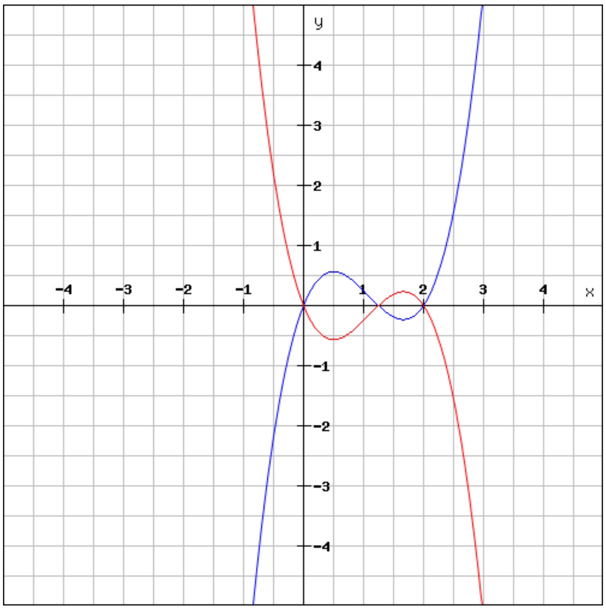


Figure 2

So the transformation has no effect on the area, and thus $A(k) = A(2-k)$.

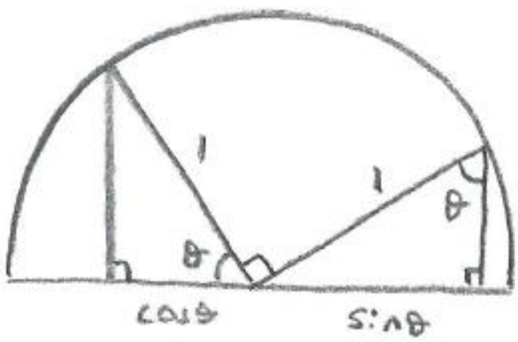
[The official solution should read " x-axis " instead of " y-axis".]

The official solution says that "as a rotation is area-preserving, then the area between the graph and the x-axis remains unchanged". It's perhaps worth adding that this depends on the rotation being 180°; ie for some other rotation the total area either side of the x-axis would not be the same.

Q4

For (i), the behaviour of the graph of $y = \frac{a+x}{a-x}$ as x approaches a can be investigated by setting $x = a - \delta$, where δ is a small positive number. This shows that $y \to \infty$ as $x \to a^-$ (meaning: x tends to a from below).

For (ii), we can draw in radii as shown below, and discover that the area between the two (identical) triangles is a quarter of the circle.



It is perhaps useful to be familiar with the generalised definition of $cos\theta$ and $sin\theta$, as the x and y coordinates of a point on the unit circle, where the radius to the point makes an angle θ with the positive x - axis.

For the alternative integration method demonstrated in the official solution, we are making the (speculative) substitution $x = sinu$, so that $dx = cosu\ du$.

For (iii), an alternative starting point is:

$(sin\theta - cos\theta)^2 \geq 0 \Rightarrow sin^2\theta + cos^2\theta - 2sin\theta cos\theta \geq 0$

$\Rightarrow 1 \geq 2sin\theta cos\theta \Rightarrow sin\theta cos\theta \leq \frac{1}{2}$

For (iv), use of the double angle formula gives:

$$\frac{area\ of\ A}{area\ of\ B} = \frac{\frac{\pi}{4}+sin\theta cos\theta}{\frac{1}{2}\pi(1)^2-\left(\frac{\pi}{4}+sin\theta cos\theta\right)} = \frac{\frac{\pi}{4}+\frac{1}{2}sin2\theta}{\frac{\pi}{4}-\frac{1}{2}sin2\theta} = \frac{\pi+2sin2\theta}{\pi-2sin2\theta}$$

which is maximised when $\theta = \frac{\pi}{4}$, to give $\frac{\pi+2}{\pi-2}$

Q5

For (ii), the official sol'n says that the answer follows 'by analogy' with (i). More justification is probably needed though (and there are obviously possibilities other than n+1 that would have given 9 for (i)).

Here there doesn't seem to be much alternative to writing out the possible outcomes, for various values of n.

This shows that, if n is odd, then the number is $\frac{n+1}{2}.2 = n + 1$, whilst if n is even it is $\frac{n}{2}.2 + 1 = n + 1$

For (iii), listing of possible combinations from scratch turns out not to yield any useful pattern. But conditioning on the hundreds digit enables the result from (ii) to be used.

It is possible to arrive at the answer by postulating that it will be a quadratic in n and substituting $n = 1,2$ & 3 in turn to give 3 simultaneous equations. This might get some credit.

The method in the official sol'n can also be set out as follows:

$\sum_{h=0}^{n}[(n-h)+1]=(n+1)(n+1)-\sum_{h=0}^{n}h$

$=(n+1)^2-\frac{1}{2}n(n+1)=\frac{1}{2}(n+1)[2(n+1)-n]$

$=\frac{1}{2}(n+1)(n+2)$

For (iv), the same method as in (iii) can also be applied:

$\sum_{h=5}^{8}[(8-h)+1]=4\times 9-\sum_{h=5}^{8}h$

$=36-(5+6+7+8)=10$

For (iv) and (v), it is however quite easy to just list the possible combinations, based on the hundreds digit.

For (vi), it turns out that the previous results are not needed.

An alternative approach for (vi) is as follows:

0 to 9: ½(9)(10)=45

10 to 19: 10+45

20 to 29: 20+45

... 90 to 99: 90+45

So far: $10\times 45+(10+20+\cdots+90)=450+10\times 45=900$

100 to 199: 100+900

200 to 299: 200+900

... 900 to 999: 900+900

Total $=10\times 900+(100+\cdots+900)$

$=9000+100\times 45=13500$

This question is a good illustration of the dilemmas to be faced for this style of question: (i) when to start listing possible combinations, and (ii) when to try to use an earlier result.