2011 MAT Paper - Q2 (2 pages; 30/8/20)

Solution

(i) $x^3 = 2x + 1 \Rightarrow x^4 = x(2x + 1) = x + 2x^2$

and $x^5 = x(x + 2x^2) = x^2 + 2(2x + 1) = 2 + 4x + x^2$

(ii) $x^{k+1} = x(A_k + B_k x + C_k x^2)$

$= A_k x + B_k x^2 + C_k x^3$

$= A_k x + B_k x^2 + C_k(2x + 1)$

$= C_k + (A_k + 2C_k)x + B_k x^2$

Also $x^{k+1} = A_{k+1} + B_{k+1}x + C_{k+1}x^2$

Equating coefficients:

$A_{k+1} = C_k$; $B_{k+1} = (A_k + 2C_k)$; $C_{k+1} = B_k$

(iii) $D_{k+1} = A_{k+1} + C_{k+1} - B_{k+1}$

$= C_k + B_k - (A_k + 2C_k)$, from (ii)

$= -C_k + B_k - A_k = -D_k$

rtp: $A_k + C_k = B_k + (-1)^k$

ie that $D_k = (-1)^k$

Now, $D_0 = A_0 + C_0 - B_0 = 1 + 0 - 0 = 1$, as $x^0 = 1$

Then $D_{k+1} = -D_k \Rightarrow D_1 = -1$; $D_2 = 1$ ...

and $D_k = (-1)^k$, as required

(iv) $F_k + F_{k+1} = A_{k+1} + C_{k+1} + A_{k+2} + C_{k+2}$ (1)

$F_{k+2} = A_{k+3} + C_{k+3} = C_{k+2} + B_{k+2}$

$= C_{k+2} + (A_{k+1} + 2C_{k+1})$

$= F_k + F_{k+1} - A_{k+2} + C_{k+1}$, from (1)

$= F_k + F_{k+1}$, as required