2010 MAT Paper - Q3 (2 pages; 28/18/20)

Solution

(i) Area $OAC <$ Area of sector OAC

$\Rightarrow \frac{1}{2}(1)^2 sinx < \frac{1}{2}(1)^2 x \Rightarrow sinx < x$

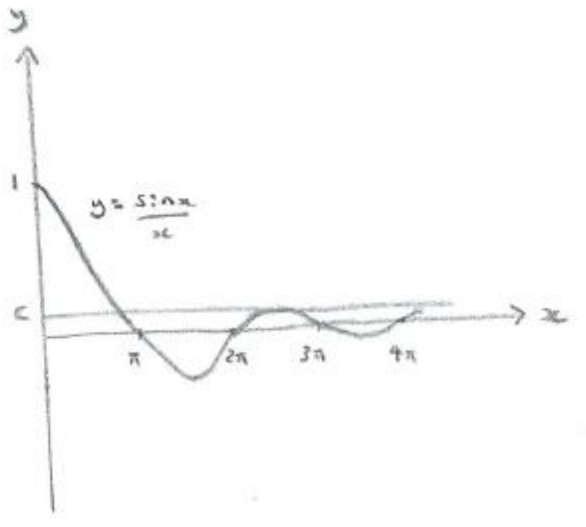
rtp: $xcosx < sinx$ or $x < tanx$ (as $0 < x < \frac{\pi}{2}$ & hence $cosx > 0$)

Area of sector $OAC <$ Area OAB

$\Rightarrow \frac{1}{2}(1)^2 x < \frac{1}{2}(1)tanx$

$\Rightarrow x < tanx$, as required.

(ii) As $x > 0$, $xcosx < sinx < x \Rightarrow cosx < \frac{sinx}{x} < 1$

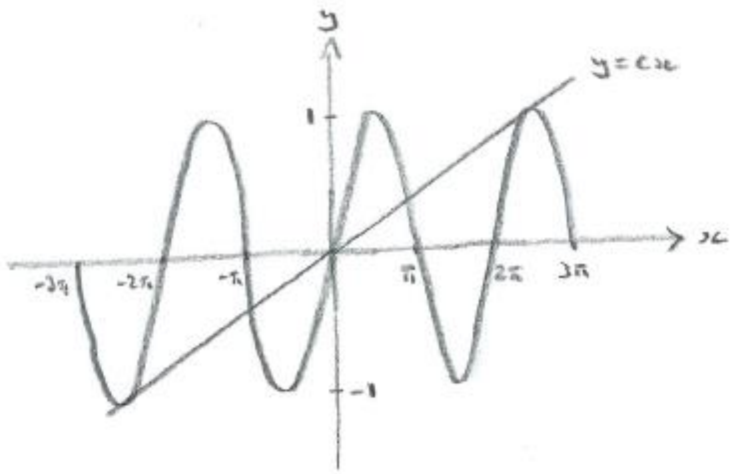


As $cosx \to 1$ as $x \to 0$, $\frac{sinx}{x}$ is trapped between 1 and a number that gets closer to 1, so that $\frac{sinx}{x} \to 1$

[L'Hôpital's rule: if $\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = 0 \; or \; \pm\infty$,

then $\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$]

(iii)



(iv) See (ii). The hump of $y = \frac{sinx}{x}$ in the diagram in (ii) between $x = 2\pi$ & $x = 3\pi$ represents the 1st positive repeated root of $\frac{sinx}{x} = c$, and therefore the 1st positive repeated root of

$sinx = cx$; ie where the graphs of $y = sinx$ & $y = cx$ touch.

(v) X is where $\frac{sinx}{x} = c$; ie the 1st positive maximum of $y = \frac{sinx}{x}$

$\frac{dy}{dx} = 0 \Rightarrow \frac{xcosx - sinx}{x^2} = 0$

$\Rightarrow xcosx - sinx = 0 \Rightarrow x = tanx$; ie $tanX = X$