2009 MAT - Q2 (3 pages; 27/8/20)

Solution

(i) $x_4 = 2x_3 - x_2 + 1 = 12 - 3 + 1 = 10$

$x_5 = 2x_4 - x_3 + 1 = 20 - 6 + 1 = 15$

(ii) $1 = A + B + C$ (1)

$3 = A + 2B + 4C$ (2)

$6 = A + 3B + 9C$ (3)

Subst. for A from (1) into (2) & (3),

$3 = (1 - B - C) + 2B + 4C \Rightarrow B + 3C = 2$ $(2a)$

$6 = (1 - B - C) + 3B + 9C \Rightarrow 2B + 8C = 5$ $(3a)$

Subst. for B from (2a) into (3a),

$2(2 - 3C) + 8C = 5 \Rightarrow 2C = 1 \Rightarrow C = \frac{1}{2}$

Then (2a) $\Rightarrow B = \frac{1}{2}$ and (1) $\Rightarrow A = 0$

(iii) [Assuming that n is supposed to be an integer; $x_{3.5}$, for example, wouldn't be defined]

To find the smallest real number satisfying $\frac{1}{2}x + \frac{1}{2}x^2 \geq 800$:

$x^2 + x - 1600 = 0 \Rightarrow x = \frac{-1+\sqrt{1+6400}}{2}$ (as $x > 0$)

The smallest integer will then be $\geq \frac{-1+\sqrt{6400}}{2} = \frac{79}{2}$,

and thus the required n is 40

[For a more rigorous proof, we could of course evaluate the quadratic for $n = 39$]

(iv) [From the fact that $\frac{x_n}{y_n}$ is supposed to have a limit, we can surmise that a quadratic expression is needed for y_n , given that x_n has a quadratic form.]

The 1st few terms for y_n are: $1, 5, 11, 19, 29, 41$

The 1st differences are $4, 6, 8, 10, 12$,

and the 2nd differences are all 2.

Therefore, y_n can be represented by a quadratic function of n, where the coefficient of n^2 is $\frac{1}{2}(2) = 1$ [this is a standard result, but we are demonstrating that the formula works]

Consider the 1st few terms for $y_n - n^2$: $0, 1, 2, 3, \ldots$

Thus $y_n - n^2 = n - 1$,

and $y_n = n^2 + n - 1$

(We know that a quadratic formula exists, and there will only be one such formula that holds for $0, 1, 2$)

[Alternative method: Let $y_n = D + En + Fn^2$, and find D, E & F as in (ii).]

$$\frac{x_n}{y_n} = \frac{\frac{1}{2}n+\frac{1}{2}n^2}{n^2+n-1} = \frac{1}{2}\left(\frac{\frac{1}{n}+1}{1+\frac{1}{n}-\frac{1}{n^2}}\right) \to \frac{1}{2}\left(\frac{1}{1}\right) = \frac{1}{2}$$

[This makes use of a university level theorem that

$lim\frac{f(n)}{g(n)} = \frac{limf(n)}{limg(n)}$, provided that $limf(n)$ & $limg(n)$ are both constants. It seems to be customary to use this theorem without further comment.]