2008 MAT - Q2 (2 pages; 27/8/20)

Solution

(i) [Pure trial and error is of course possible]

${x_1}^2 = 1 + 2{y_1}^2$, so that x_1 is odd

${y_1}^2 = \frac{{x_1}^2-1}{2} = \frac{(x_1-1)(x_1+1)}{2} = \frac{\text{even}\times\text{even}}{2} = \text{even}$

Try $y_1 = 2$, giving $x_1 = 3$

(ii) $(3x_n + 4y_n)^2 - 2(ax_n + by_n)^2 = {x_n}^2 - 2{y_n}^2$

$\Rightarrow 9{x_n}^2 + 16{y_n}^2 + 24x_ny_n - 2a^2{x_n}^2 - 2b^2{y_n}^2 - 4abx_ny_n$

$= {x_n}^2 - 2{y_n}^2$

As this is to be true for all x_n& y_n, equating coefficients:

${x_n}^2: 9 - 2a^2 = 1 \quad (1)$

${y_n}^2: 16 - 2b^2 = -2 \quad (2)$

$x_ny_n: 24 - 4ab = 0 \quad (3)$

From (1) & (2), $a = 2$ & $b = 3$, and these values satisfy (3).

(iii) If $x_1 = 3$ & $y_1 = 2$ (from (i)),

then, with the a & b just found,

$x_2 = 3x_1 + 4y_1 = 17$ & $y_2 = 2x_1 + 3y_1 = 12$

$x_3 = 3x_2 + 4y_2 = 99$ & $y_3 = 2x_2 + 3y_2 = 70$

So, from (2), ${x_3}^2 - 2{y_3}^2 = {x_2}^2 - 2{y_2}^2 = {x_1}^2 - 2{y_1}^2 = 1$;

and hence $X^2 - 2Y^2 = 1$, where $X = 99$ & $Y = 70$

Check: $99^2 - 2(70^2) = (100-1)^2 - 2(4900)$

$= 10000 + 1 - 200 - 9800 = 1$

(iv) $\frac{x_{n+1}}{y_{n+1}} = \frac{3x_n+4y_n}{2x_n+3y_n} = \frac{3\left(\frac{x_n}{y_n}\right)+4}{2\left(\frac{x_n}{y_n}\right)+3}$

in the limit as $\frac{x_n}{y_n} \to L$ (so that $\frac{x_{n+1}}{y_{n+1}} \to L$ also),

$L = \frac{3L+4}{2L+3} \Rightarrow 2L^2 + 3L = 3L + 4 \Rightarrow L^2 = 2$ and $L = \sqrt{2}$

(As x_1& y_1 are both positive, the expressions for x_{n+1}& y_{n+1} imply that subsequent x_n& y_n are also positive, and hence L is positive.)