Logarithms – Exercises (with Sol'ns) (11 pages ; 1/2/26)

Show that $ln\left(x-\sqrt{x^2-1}\right) = -ln\left(x+\sqrt{x^2-1}\right)$

Show that $ln\left(x-\sqrt{x^2-1}\right) = -ln\left(x+\sqrt{x^2-1}\right)$

Solution

$-ln\left(x+\sqrt{x^2-1}\right) = ln\left(\frac{1}{x+\sqrt{x^2-1}}\right)$

$= ln\ (\frac{1}{x+\sqrt{x^2-1}}\ .\frac{x-\sqrt{x^2-1}}{x-\sqrt{x^2-1}})$

$= ln\ (\frac{x-\sqrt{x^2-1}}{1})$

If $k = log_{24}12$, write the following in terms of k:

(a) $log_{24}2$ (b) $log_{24}6$

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Solution

(a) $log_{24}2 = log_{24}\left(\frac{24}{12}\right) = log_{24}24 - log_{24}12 = 1 - k$

(b) $log_{24}6 = log_{24}\left(\frac{12}{2}\right) = log_{24}12 - log_{24}2 = k - (1 - k) =$
$2k - 1$

[or $log_{24}6 = log_{24}\left(\frac{24}{4}\right) = log_{24}24 - log_{24}4 = 1 - log_{24}(2^2)$

Prove that $log_b c = \frac{log_a c}{log_a b}$

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Solution

Method 1

rtp $log_a b \, log_b c = log_a c$ (*)

Let $b = a^x$ & $c = b^y$

Then $c = (a^x)^y = a^{xy}$

and $log_a c = xy = log_a b \, log_b c$, as required

Method 2

(*) is equivalent to $a^{log_a b \, log_b c} = a^{log_a c}$ (as $y = a^x$ is an increasing function)

ie $(a^{log_a b})^{log_b c} = c$ (**)

and the LHS equals $b^{log_b c} = c$, so that (**) holds, and hence (*) holds as well

Method 3 (informal)

To show that $log_a b . log_b c = log_a c$:

In terms of powers, p takes you from a to b, and q takes you from b to c; so pq takes you from a to c

Prove that $\int \frac{1}{x} dx = \ln|x|$ for all $x \neq 0$,

assuming that $\int \frac{1}{x} dx = \ln x$ for $x > 0$

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Solution

Method 1

If $\int \frac{1}{x} dx = \ln x$ for $x > 0$, then $\frac{d}{dx}(lnx) = \frac{1}{x}$ for $x > 0$

For the case where $x < 0$:

Let $y = -x$, so that $\frac{d}{dy}(lny) = \frac{1}{y}$, as $y > 0$

[To convert back to xs:]

Hence $\frac{d}{dx}(lny).\frac{dx}{dy} = \frac{1}{(-x)}$

giving $\frac{d}{dx}(\ln[-x])(-1) = \frac{1}{(-x)}$

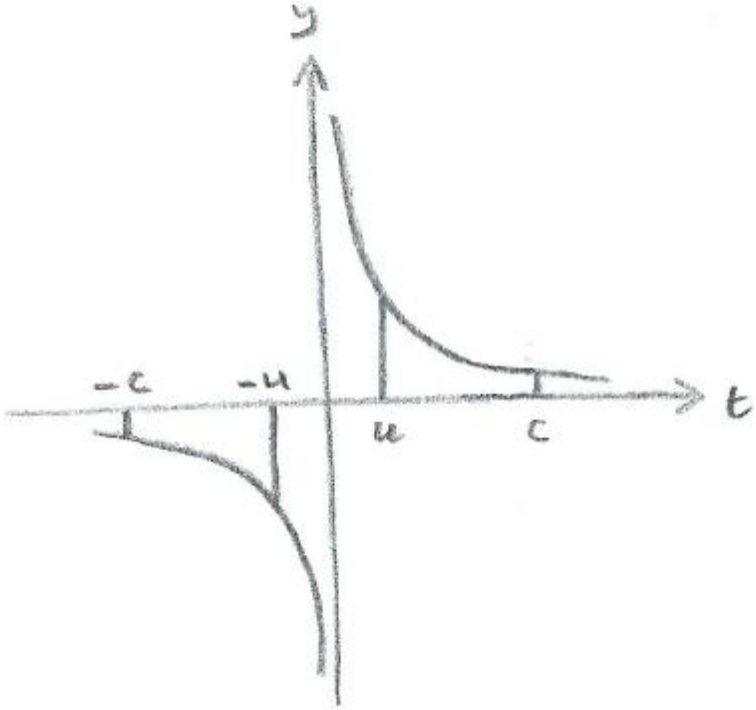
and so $\frac{d}{dx}(\ln|x|) = \frac{1}{x}$ for $x < 0$ (*)

and therefore $\int \frac{1}{x} dx = \ln|x|$ for $x < 0$, as well as $x > 0$

[Note that the function $y = \ln$ |x| for $x < 0$ is the reflection in the y-axis of $y = \ln x$ $(for\ x > 0)$, and therefore has a negative gradient, which agrees with (*).]

Method 2

Referring to the diagram below, where $u = -x > 0$ & $c > 0$,



$\int_{-c}^{x} \frac{1}{t} dt = \int_{-c}^{-u} \frac{1}{t} dt$

$=\ -$ (positive) area between graph and t-axis on LHS

$=\ -$ (positive) area between graph and t-axis on RHS

$= -\int_{u}^{c} \frac{1}{t} dt = \int_{c}^{u} \frac{1}{t} dt = lnu - lnc$

As $\int \frac{1}{x} dx$ only differs from $\int_{-c}^{x} \frac{1}{t} dt$ by an arbitrary constant, it follows that, when $x < 0$, $\int \frac{1}{x} dx = ln\, u + C = \ln|-x| + C$, as required.

Write $log_2 3$ in terms of logs to the base 10

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Solution

Method 1

Standard result: $log_a b\ log_b c = log_a c$

[a is raised to the power of $log_a c$ in order to get to c; alternatively, raise a to the power of $p = log_a b$, to get to b, and then raise b to the power of $q = log_b c$, to get to c; thus $a^p = b\ \ and\ b^q = c$, which gives $(a^p)^q = c$, and hence $a^{pq} = c$, so that $log_a c = pq = log_a b\ log_b c$]

Then $log_b c = \frac{log_{10} c}{log_{10} b}$, so that $log_2 3 = \frac{log_{10} 3}{log_{10} 2}$

Method 2

Set up an equation, as follows:

Let $log_2 3 = x$

[The advantage of creating an equation is that we then have something that can be manipulated.]

$\Rightarrow 3 = 2^x$

$\Rightarrow log_{10} 3 = x log_{10} 2$

$\Rightarrow log_2 3 = x = \frac{log_{10} 3}{log_{10} 2}$