

Linear Systems of Differential Equations (7 pages; 14/9/19)

(1) Example

$$\frac{dx}{dt} = 4x - 6y - 9\sin t \quad (1)$$

$$\frac{dy}{dt} = 3x - 5y - 7\sin t \quad (2)$$

The aim is to find both x and y (the **dependent** variables) as functions of t (the **independent** variable).

To do this we can differentiate (1) wrt t , to obtain a 2nd order equation for x , and then use (2) and (1) to substitute for $\frac{dy}{dt}$.

$$\text{Thus, (1)} \Rightarrow \frac{d^2x}{dt^2} = 4\frac{dx}{dt} - 6\frac{dy}{dt} - 9\cos t \quad (3)$$

$$\text{Also (2)} \Rightarrow 6\frac{dy}{dt} = 18x - 30y - 42\sin t$$

$$\text{and then (1)} \Rightarrow 6\frac{dy}{dt} = 18x - 5(4x - 9\sin t - \frac{dx}{dt}) - 42\sin t \quad (4)$$

Substituting from (4) into (3) gives:

$$\frac{d^2x}{dt^2} = 4\frac{dx}{dt} - (18x - 5(4x - 9\sin t - \frac{dx}{dt}) - 42\sin t) - 9\cos t$$

$$\text{which gives } \frac{d^2x}{dt^2} + \frac{dx}{dt} - 2x = -3\sin t - 9\cos t \quad (5)$$

[This approach can be remembered by thinking of the 2nd order equation in x that we are aiming for. This makes the 1st step clear: obtain $\frac{d^2x}{dt^2}$ by differentiating the expression for $\frac{dx}{dt}$, and then eliminate the unwanted $\frac{dy}{dt}$ from the 2nd original equation, and then the resulting unwanted y from the 1st original equation.]

This is then solved to give a general solution for $x(t)$, and a general solution for $y(t)$ can be found by substituting for $x(t)$ and $\frac{dx}{dt}$ in (1), having first differentiated $x(t)$ to give $\frac{dx}{dt}$.

In this example, (5) produces the general solution

$$x(t) = Ae^t + Be^{-2t} + 3\cos t \quad (6)$$

$$\text{Then } \frac{dx}{dt} = Ae^t - 2Be^{-2t} - 3\sin t$$

Substituting into (1) then gives

$$Ae^t - 2Be^{-2t} - 3\sin t = 4(Ae^t + Be^{-2t} + 3\cos t) - 6y - 9\sin t$$

$$\text{which leads to } y = \frac{A}{2}e^t + Be^{-2t} + 2\cos t - \sin t \quad (7)$$

[Note: Although an expression for y could be obtained by the same method that produced the expression for x , we wouldn't be able to obtain the relation between the arbitrary constants of x and y .]

Equations (6) & (7) are parametric equations for x and y . In simple situations, it may be possible to eliminate t , to give a relationship between x and y . The resulting graph of y against x is called the **solution curve**.

The system may approach an equilibrium position as $t \rightarrow \infty$.

Alternative approach

If we are choosing to eliminate y , make y the subject of the 1st equation (so that y is a function of x & t); then differentiate the resulting expression for y , to obtain an expression for $\frac{dy}{dt}$ (in terms of x & t). These expressions for y and $\frac{dy}{dt}$ can then be substituted

into the 2nd original equation, to obtain a 2nd order equation in x .

In this case:

$$\text{From (1), } y = \frac{1}{6}(4x - 9\sin t - \frac{dx}{dt})$$

$$\text{Then } \frac{dy}{dt} = \frac{1}{6}(4\frac{dx}{dt} - 9\cos t - \frac{d^2x}{dt^2})$$

and substituting these expressions into (2) gives:

$$\frac{1}{6}\left(4\frac{dx}{dt} - 9\cos t - \frac{d^2x}{dt^2}\right) = 3x - \frac{5}{6}(4x - 9\sin t - \frac{dx}{dt}) - 7\sin t,$$

$$\Rightarrow 4\frac{dx}{dt} - 9\cos t - \frac{d^2x}{dt^2} = 18x - 20x + 45\sin t + 5\frac{dx}{dt} - 42\sin t$$

$$\Rightarrow \frac{d^2x}{dt^2} + \frac{dx}{dt} - 2x = -3\sin t - 9\cos t, \text{ as before}$$

(2) Predator-prey model

This commonly take the form of a pair of linear equations, such as

$$\frac{dx}{dt} = ax + by, \quad \frac{dy}{dt} = cy - ex, \text{ with } a, b, c \text{ \& } e > 0$$

(x is the population of the predator; its growth rate increases with the size of its own population (due to breeding), and with the size of the prey population (which provides food); the growth rate of the prey population y increases with the size of its own population (due to breeding), and reduces as the predator population increases.)

Exercise 1

The following pair of differential equations is to be solved:

$$\frac{dx}{dt} = ax + by + f(t) \quad (1), \quad \frac{dy}{dt} = cx + ey + g(t) \quad (2)$$

Show that the complementary functions obtained for x and y are the same.

Solution

Choosing to eliminate x , (2) $\Rightarrow \frac{d^2y}{dt^2} = c \frac{dx}{dt} + e \frac{dy}{dt} + g'(t)$ (3)

Then, from (1), $\frac{d^2y}{dt^2} = c(ax + by + f(t)) + e \frac{dy}{dt} + g'(t)$

and, from (2),

$$\frac{d^2y}{dt^2} = a\left(\frac{dy}{dt} - ey - g(t)\right) + cby + cf(t) + e \frac{dy}{dt} + g'(t),$$

so that $\frac{d^2y}{dt^2} - (a + e) \frac{dy}{dt} + (ae - bc)y = cf(t) - ag(t) + g'(t)$

Choosing instead to eliminate y , (1) $\Rightarrow \frac{d^2x}{dt^2} = a \frac{dx}{dt} + b \frac{dy}{dt} + f'(t)$

Then, from (2), $\frac{d^2x}{dt^2} = a \frac{dx}{dt} + b(cx + ey + g(t)) + f'(t)$

and, from (1),

$$\frac{d^2x}{dt^2} = a \frac{dx}{dt} + bcx + e \left(\frac{dx}{dt} - ax - f(t) \right) + bg(t) + f'(t)$$

so that $\frac{d^2x}{dt^2} - (a + e) \frac{dx}{dt} + (ae - bc)x = -ef(t) + bg(t) + f'(t)$

As the auxiliary equations are the same for the two 2nd order equations, they have the same complementary functions.

Exercise 2

The following pair of equations models the populations of two competing species, at time t .

$$100 \frac{dx}{dt} = 2x - 12y, \quad 100 \frac{dy}{dt} = y - x$$

- (i) Find the general solution of the equations.
- (ii) Initially there are 700 animals of each species. Find expressions for the numbers of each species at time t .
- (iii) Determine whether either species will die out.
- (iv) Investigate different starting values to see whether extinction is inevitable.

Solution

(i) [Using the 'alternative approach']

To eliminate x , make x the subject of the 2nd equation, to give

$$x = y - 100 \frac{dy}{dt}$$

Then differentiate to give $\frac{dx}{dt} = \frac{dy}{dt} - 100 \frac{d^2y}{dt^2}$

Substituting these expressions for x and $\frac{dx}{dt}$ into the 1st equation

$$\text{then gives } 100 \left(\frac{dy}{dt} - 100 \frac{d^2y}{dt^2} \right) = 2 \left(y - 100 \frac{dy}{dt} \right) - 12y$$

$$\text{or } 10000 \frac{d^2y}{dt^2} - 300 \frac{dy}{dt} - 10y = 0$$

$$\text{or } 1000 \frac{d^2y}{dt^2} - 30 \frac{dy}{dt} - y = 0$$

The auxiliary equation is $1000\lambda^2 - 30\lambda - 1 = 0$

$$\Rightarrow \lambda = \frac{30 \pm \sqrt{900 + 4000}}{2000} = \frac{3 \pm 7}{200} = \frac{1}{20} \text{ or } -\frac{4}{200}; \text{ ie } 0.05 \text{ or } -0.02$$

Hence $y = Ae^{0.05t} + Be^{-0.02t}$

Then x can be obtained from the rearranged 2nd equation

$x = y - 100 \frac{dy}{dt}$, as follows:

$$\begin{aligned} x &= Ae^{0.05t} + Be^{-0.02t} - 100(0.05Ae^{0.05t} - 0.02Be^{-0.02t}) \\ &= -4Ae^{0.05t} + 3Be^{-0.02t} \end{aligned}$$

(ii) When $t = 0$, $x = 700$ & $y = 700$, so that

$$700 = -4A + 3B \text{ and } 700 = A + B,$$

$$\text{Then } 700 = -4A + 3(700 - A),$$

$$\text{so that } 7A = 1400 \text{ and } A = 200; B = 500$$

$$\text{Thus } x = -800e^{0.05t} + 1500e^{-0.02t}$$

$$\text{and } y = 200e^{0.05t} + 500e^{-0.02t}$$

(iii) $y > 0$ for all t and so will not become extinct

$$x = 0 \Rightarrow -800e^{0.05t} + 1500e^{-0.02t} = 0$$

$$\Rightarrow \frac{15}{8} = e^{0.07t}$$

$$\Rightarrow t = \frac{1}{0.07} \ln\left(\frac{15}{8}\right) = 8.980,$$

ie species x will become extinct in approximately 9 years

$$\text{(iv) } x = -4Ae^{0.05t} + 3Be^{-0.02t} = 0 \Rightarrow \frac{3B}{4A} = e^{0.07t}$$

So species x will survive whenever A and B have opposite signs (as $x = 0$ is not possible then).

Let initial populations of x & y be x_0 & y_0 .

$$\text{Then } x_0 = -4A + 3B \text{ and } y_0 = A + B$$

$$\Rightarrow x_0 = -4A + 3(y_0 - A)$$

$$\text{and hence } A = -\frac{1}{7}(x_0 - 3y_0) = \frac{1}{7}(3y_0 - x_0)$$

$$\text{and } B = y_0 - \frac{1}{7}(3y_0 - x_0) = \frac{1}{7}(4y_0 + x_0)$$

If $A < 0$ & $B > 0$, then $3y_0 - x_0 < 0$ and $4y_0 + x_0 > 0$,

so that $y_0 < \frac{1}{3}x_0$ (with the 2nd inequality always holding)

If $A > 0$ & $B < 0$, then $3y_0 - x_0 > 0$ and $4y_0 + x_0 < 0$,

but the 2nd inequality cannot be satisfied.

So species x will survive whenever $y_0 < \frac{1}{3}x_0$

(in the original case, $700 \not< \frac{1}{3}(700)$, and so x becomes extinct).