Linear Systems of Differential Equations (7 pages; 9/3/21)

(1) Example

$\frac{dx}{dt} = 4x - 6y - 9sint$ (1)

$\frac{dy}{dt} = 3x - 5y - 7sint$ (2)

The aim is to find both x and y (the **dependent** variables) as functions of t (the **independent** variable).

To do this we can differentiate (1) wrt t, to obtain a 2nd order equation for x, and then use (2) and (1) to substitute for $\frac{dy}{dt}$.

Thus, (1)$\Rightarrow \frac{d^2x}{dt^2} = 4\frac{dx}{dt} - 6\frac{dy}{dt} - 9cost$ (3)

Also (2)$\Rightarrow 6\frac{dy}{dt} = 18x - 30y - 42sint$

and then (1) $\Rightarrow 6\frac{dy}{dt} = 18x - 5(4x - 9sint - \frac{dx}{dt}) - 42sint$ (4)

Substituting from (4) into (3) gives:

$$\frac{d^2x}{dt^2} = 4\frac{dx}{dt} - (18x - 5(4x - 9sint - \frac{dx}{dt}) - 42sint) - 9cost$$

which gives $\frac{d^2x}{dt^2} + \frac{dx}{dt} - 2x = -3sint - 9cost$ (5)

[This approach can be remembered by thinking of the 2nd order equation in x that we are aiming for. This makes the 1st step clear: obtain $\frac{d^2x}{dt^2}$ by differentiating the expression for $\frac{dx}{dt}$, and then eliminate the unwanted $\frac{dy}{dt}$ from the 2nd original equation, and then the resulting unwanted y from the 1st original equation.]

This is then solved to give a general solution for $x(t)$, and a general solution for $y(t)$ can be found by substituting for $x(t)$ and $\frac{dx}{dt}$ in (1), having first differentiated $x(t)$ to give $\frac{dx}{dt}$.

In this example, (5) produces the general solution

$x(t) = Ae^{t} + Be^{-2t} + 3cost$ (6)

Then $\frac{dx}{dt} = Ae^{t} - 2Be^{-2t} - 3sint$

Substituting into (1) then gives

$Ae^{t} - 2Be^{-2t} - 3sint = 4(Ae^{t} + Be^{-2t} + 3cost) - 6y - 9sint$

which leads to $y = \frac{A}{2}e^{t} + Be^{-2t} + 2cost - sint$ (7)

[Note: Although an expression for y could be obtained by the same method that produced the expression for x, we wouldn't be able to obtain the relation between the arbitrary constants of x and y.]

Equations (6) & (7) are parametric equations for x and y. In simple situations, it may be possible to eliminate t, to give a relationship between x and y. The resulting graph of y against x is called the **solution curve**.

The system may approach an equilibrium position as $t \to \infty$.

Alternative approach

If we are choosing to eliminate y, make y the subject of the 1st equation (so that y is a function of x & t) ; then differentiate the resulting expression for y, to obtain an expression for $\frac{dy}{dt}$ (in terms of x & t). These expressions for y and $\frac{dy}{dt}$ can then be substituted into the 2nd original equation, to obtain a 2nd order equation in x.

In this case:

From (1), $y = \frac{1}{6}(4x - 9sint - \frac{dx}{dt})$

Then $\frac{dy}{dt} = \frac{1}{6}(4\frac{dx}{dt} - 9cost - \frac{d^2x}{dt^2})$

and substitituting these expressions into (2) gives:

$\frac{1}{6}\left(4\frac{dx}{dt} - 9cost - \frac{d^2x}{dt^2}\right) = 3x - \frac{5}{6}(4x - 9sint - \frac{dx}{dt}) - 7sint,$

$\Rightarrow 4\frac{dx}{dt} - 9cost - \frac{d^2x}{dt^2} = 18x - 20x + 45sint + 5\frac{dx}{dt} - 42sint$

$\Rightarrow \frac{d^2x}{dt^2} + \frac{dx}{dt} - 2x = -3sint - 9cost$, as before

(2) Predator-prey model

This commonly take the form of a pair of linear equations, such as $\frac{dx}{dt} = ax + by,\ \frac{dy}{dt} = cy - ex$, with $a, b, c\ \&\ e > 0$

(x is the population of the predator; its growth rate increases with the size of its own population (due to breeding), and with the size of the prey population (which provides food); the growth rate of the prey population y increases with the size of its own population (due to breeding), and reduces as the predator population increases.)

Exercise 1

The following pair of differential equations is to be solved:

$\frac{dx}{dt} = ax + by + f(t)$ (1), $\frac{dy}{dt} = cx + ey + g(t)$ (2)

Show that the complementary functions obtained for x and y are the same.

Solution

Choosing to eliminate x, (2) $\Rightarrow \frac{d^2y}{dt^2} = c\frac{dx}{dt} + e\frac{dy}{dt} + g'(t)$ (3)

Then, from (1), $\frac{d^2y}{dt^2} = c(ax + by + f(t)\ ) + e\frac{dy}{dt} + g'(t)$

and, from (2),

$\frac{d^2y}{dt^2} = a(\frac{dy}{dt} - ey - g(t)) + cby + cf(t)\ ) + e\frac{dy}{dt} + g'(t)$,

so that $\frac{d^2y}{dt^2} - (a + e)\frac{dy}{dt} + (ae - bc)y = cf(t) - ag(t) + g'(t)$

Choosing instead to eliminate y, (1) $\Rightarrow \frac{d^2x}{dt^2} = a\frac{dx}{dt} + b\frac{dy}{dt} + f'(t)$

Then, from (2), $\frac{d^2x}{dt^2} = a\frac{dx}{dt} + b(cx + ey + g(t)) + f'(t)$

and, from (1),

$\frac{d^2x}{dt^2} = a\frac{dx}{dt} + bcx + e\left(\frac{dx}{dt} - ax - f(t)\right) + bg(t) + f'(t)$

so that $\frac{d^2x}{dt^2} - (a + e)\frac{dx}{dt} + (ae - bc)x = -ef(t) + bg(t) + f'(t)$

As the auxiliary equations are the same for the two 2nd order equations, they have the same complementary functions.

Exercise 2

The following pair of equations models the populations of two competing species, at time t.

$$100\frac{dx}{dt} = 2x - 12y, \;\; 100\frac{dy}{dt} = y - x$$

(i) Find the general solution of the equations.

(ii) Initially there are 700 animals of each species. Find expressions for the numbers of each species at time t.

(iii) Determine whether either species will die out.

(iv) Investigate different starting values to see whether extinction is inevitable.

Solution

(i) [Using the 'alternative approach']

To eliminate x, make x the subject of the 2nd equation, to give

$$x = y - 100\frac{dy}{dt}$$

Then differentiate to give $\frac{dx}{dt} = \frac{dy}{dt} - 100\frac{d^2y}{dt^2}$

Substituting these expressions for x and $\frac{dx}{dt}$ into the 1st equation then gives $100(\frac{dy}{dt} - 100\frac{d^2y}{dt^2}) = 2(y - 100\frac{dy}{dt}) - 12y$

or $10000\frac{d^2y}{dt^2} - 300\frac{dy}{dt} - 10y = 0$

or $1000\frac{d^2y}{dt^2} - 30\frac{dy}{dt} - y = 0$

The auxiliary equation is $1000\lambda^2 - 30\lambda - 1 = 0$

$\Rightarrow \lambda = \frac{30\pm\sqrt{900+4000}}{2000} = \frac{3\pm7}{200} = \frac{1}{20}\ or - \frac{4}{200}$; ie $0.05\ or - 0.02$

Hence $y = Ae^{0.05t} + Be^{-0.02t}$

Then x can be obtained from the rearranged 2nd equation

$x = y - 100\frac{dy}{dt}$, as follows:

$x = Ae^{0.05t} + Be^{-0.02t} - 100(0.05Ae^{0.05t} - 0.02Be^{-0.02t})$

$= -4Ae^{0.05t} + 3Be^{-0.02t}$

(ii) When $t = 0, x = 700$ & $y = 700$, so that

$700 = -4A + 3B$ and $700 = A + B$,

Then $700 = -4A + 3(700 - A)$,

so that $7A = 1400$ and $A = 200$; $B = 500$

Thus $x = -800e^{0.05t} + 1500e^{-0.02t}$

and $y = 200e^{0.05t} + 500e^{-0.02t}$

(iii) $y > 0$ for all t and so will not become extinct

$x = 0 \Rightarrow -800e^{0.05t} + 1500e^{-0.02t} = 0$

$\Rightarrow \frac{15}{8} = e^{0.07t}$

$\Rightarrow t = \frac{1}{0.07}\ln\left(\frac{15}{8}\right) = 8.980$,

ie species x will become extinct in approximately 9 years

(iv) $x = -4Ae^{0.05t} + 3Be^{-0.02t} = 0 \Rightarrow \frac{3B}{4A} = e^{0.07t}$

So species x will survive if A and B have opposite signs (as $x = 0$ is not possible then, as $e^{0.07t} > 0$). But $A > 0, B < 0$ is not possible, as then $x = -4Ae^{0.05t} + 3Be^{-0.02t} < 0$), so the conclusion is that species x will survive if $A < 0$ & $B > 0$.

Let the initial populations of x & y be x_0 & y_0.

Then $x_0 = -4A + 3B$ and $y_0 = A + B$

$\Rightarrow x_0 = -4A + 3(y_0 - A)$

and hence $A = -\frac{1}{7}(x_0 - 3y_0) = \frac{1}{7}(3y_0 - x_0)$

and $B = y_0 - \frac{1}{7}(3y_0 - x_0) = \frac{1}{7}(4y_0 + x_0)$

If $A < 0$ & $B > 0$, then $3y_0 - x_0 < 0$ and $4y_0 + x_0 > 0$,

so that $y_0 < \frac{1}{3}x_0$ (with the 2nd inequality always holding)

So species x will survive whenever $y_0 < \frac{1}{3}x_0$

(in the original case, $700 \nless \frac{1}{3}(700)$, and so x becomes extinct).