Linear Programming - Exercises (Sol'ns)

(13 pages; 16/8/19)

(1) Maximise $\text{P} = 2x + 3y$

subject to $x + 2y \leq 12$

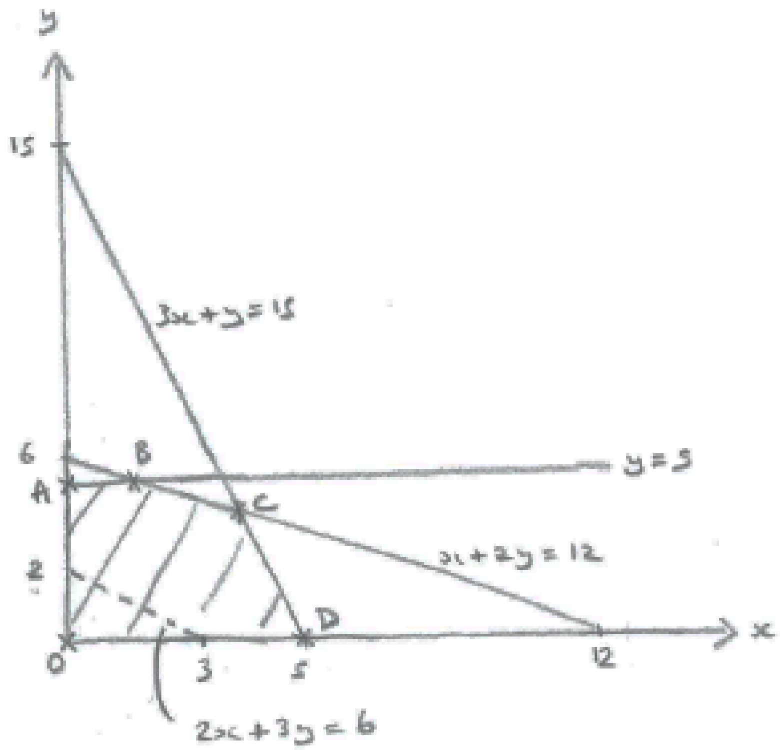
$3x + y \leq 15$

$y \leq 5$

$x \geq 0$, $y \geq 0$

Integer values of x & y are required.

Solution



By considering lines parallel to $2x + 3y = 6$ (for example), C is the optimal solution, if x& y are allowed to take non-integer values.

At C:

$x + 2y = 12$ (1)

$3x + y = 15$ (2)

$x + 2y = 12$ (1)

$6x + 2y = 30$ (3)= 2×(2)

$(3) - (1) \Rightarrow 5x = 18 \Rightarrow x = \frac{18}{5} = 3.6$

Then $(2) \Rightarrow y = 15 - \frac{54}{5} = \frac{21}{5} = 4.2$

$\Rightarrow P = 7.2 + 12.6 = 19.8$

Check:

At A, P=15

At B, P=19

At D, P=10

(confirming that C is the optimal vertex).

To find integer solutions, consider neighbouring points:

(3,4): $x + 2y = 11$, $3x + y = 13$, $P = 18$

(3,5): $x + 2y = 13$ (reject)

(4,4): $x + 2y = 12$, $3x + y = 16$ (reject)

(4,5): $x + 2y = 14$ (reject)

Thus a good solution is $x = 3, y = 4$, when $P = 18$.

However, note that if we consider the point (2,5):

$x + 2y = 12, 3x + y = 11,\ P = 19$

Thus, we cannot guarantee to find the optimal solution by the above method.

(2) A company makes sofas and upholstered chairs. Each sofa requires $1m^3$ of material and 14 hours of labour to make, and sells for a profit of £200. Each chair requires $0.2m^3$ of material and 4 hours of labour to make, and sells for a profit of £30. Given that $50m^3$ of material and 840 hours of labour are available, use Linear Programming to find the number of sofas and chairs that are required, in order to optimise profit, commenting on your answer.

Solution

The objective function to be maximised is:

$P = 200s + 30c$, where s and c are the numbers of sofas and chairs made;

subject to the following constraints:

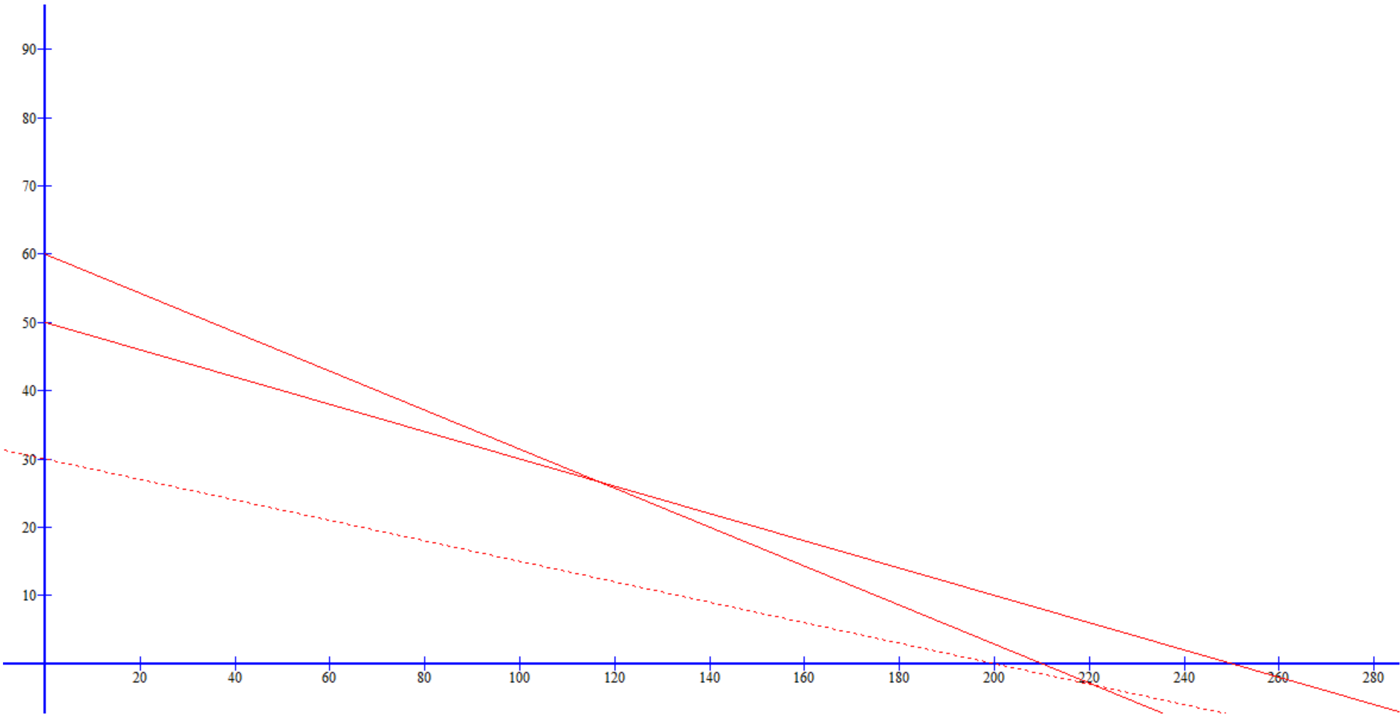
$s + 0.2c \leq 50$ or $5s + c \leq 250$

$14s + 4c \leq 840$ or $7s + 2c \leq 420$

$t, c \geq 0$ (integers)

[For exam purposes, you may wish to use a letter other than s, to avoid confusion with the number 5.]

The diagram (with s against c) shows the constraint lines and feasible region, as well as the (dotted) line $6000 = 200s + 30c$, which is parallel to the objective function.



As the gradient of the line representing the objective function is close to that of one of the constraint lines, it may not be clear which of the vertices of the feasible region maximises P (ie which vertex is the last to be passed through, as the objective function moves away from the Origin). Instead we can determine the value of P at the two likely vertices:

At (0,50) [A], $P = 200(50) = 10000$

At the intersection of the constraint lines [B],

$5s + c = 250$ and $7s + 2c = 420$,

so that $10s + 2c = 500$ and $3s = 80$; $s = \frac{80}{3}$ and $c = 250 - 5\left(\frac{80}{3}\right) = \frac{350}{3}$

Then $P = 200\left(\frac{80}{3}\right) + 30\left(\frac{350}{3}\right) = \frac{26500}{3} = 8833$

So P is maximised at A, where $s = 50$ and $c = 0$, so that 50 sofas and no chairs should be made.

[Had the optimal solution occurred at B, an integer solution would need to be found.]

However, this may not be a sensible solution for the company, for the following reasons:

- customers may wish to buy chairs to go with their sofa

- customers may be disappointed at the lack of chairs (when they are not buying a sofa)

- the company might prefer to maintain its capacity to make chairs

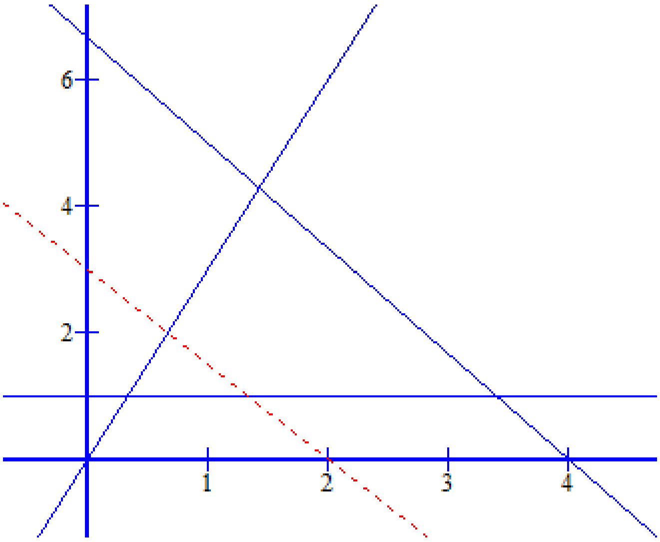
(3) Solve the following Linear Programming problem:

Minimise $P = 3x + 2y$,

subject to $5x + 3y \geq 20$
$y \leq 3x$
$x \geq 0, y \geq 1$; x & y integers

Solution

The diagram shows the constraint lines, as well as the (dotted) line
$3x + 2y = 6$, which is parallel to the objective function.



As the line representing the objective function moves away from the Origin, it first enters the feasible region at the intersection of $5x + 3y = 20$ and

$y = 1$; ie at the vertex $(3\frac{2}{5}, 1)$.

To establish an integer solution, consider first of all $x = 3$:

Then $P = 3x + 2y$, and $5x + 3y \geq 20$, $y \leq 3x$ & $y \geq 1$

So $3y \geq 5$; ie $y \geq \frac{5}{3}$, and $y \leq 9$, and $y \geq 1$,

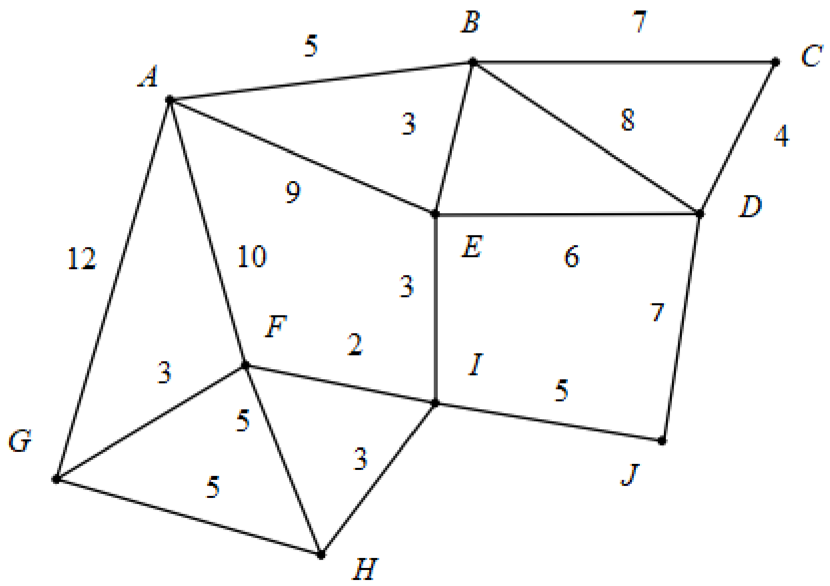
so that $y = 2$ minimises $P = 3x + 2y$, with $P = 13$.

Then consider $x = 4$, which gives $3y \geq 0$ and $y \leq 12$ & $y \geq \mathbf{1}$ so that $y = 1$ minimises $P = 3x + 2y$, with $P = 14$.

Thus the required integer solution is (3,2), with $P = 13$.

[Strictly speaking, this is just a good integer solution, as it is (just) possible that the best integer solution lies elsewhere in the feasible region.]

(4) It is required to find the shortest distance between A and J in the network below. Formulate this as a linear programming problem.



Solution

With AB, AE etc being binary variables, where $AB = 1$ means that the arc AB is travelled along:

Minimise $P = 5AB + 9AE + 10AF + 12AG + 7BC + 7CB + 8BD + 8DB + 3BE + 3EB + 4CD + 4DC + 6DE + 6ED + 7DJ + 3EI + 3IE + 3FG + 3GF + 2FI + 2IF + 5FH + 5HF + 5GH + 5HG + 3HI + 3IH + 5IJ$

[Arcs not involving A or J can be travelled along in either direction, and so are duplicated.]

$AB + AE + AF + AG = 1$ [the path has to pass along just one of the arcs leading from A]

$DJ + IJ = 1$ [the path has to pass along just one of the arcs leading to J]

$AB + EB + DB + CB = BE + BD + BC$

[if we enter B, then we must leave it - each side will total either 0 or 1]

$BC + DC = CB + CD$ [similarly for C]

$BD + CD + ED = DB + DC + DE + DJ$ [D]

$AE + BE + DE + IE = EB + ED + EI$ [E]

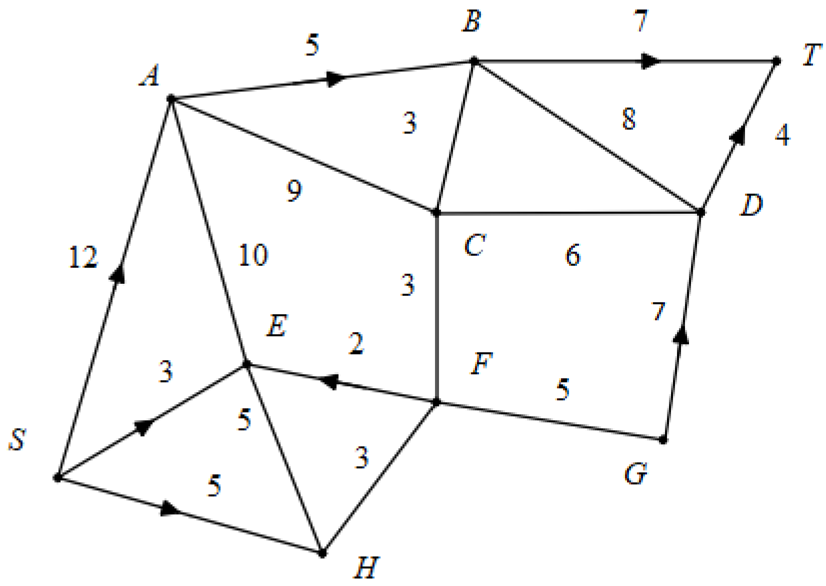
$AF + GF + HF + IF = FG + FH + FI$ [F]

$AG + FG + HG = \text{G}F + GH$ [G]

$FH + GH + IH = HF + HG + HI$ [H]

$EI + FI + HI = IE + IF + IH + IJ$ [I]

(5) The network below shows the maximum capacity for each arc. It is required to maximise the flow across the network, from S to T. Formulate this as a linear programming problem.



Solution

With SA, AB etc being non-negative integers, representing the flows along the arcs:

Maximise P = SA + SE + SH, subject to the following constraints:

$SA \leq 12, SE \leq 3, SH \leq 5$
$AB \leq 5, AC \leq 9, CA \leq 9, AE \leq 10, EA \leq 10$
$BC \leq 3, CB \leq 3, BD \leq 8, DB \leq 8, BT \leq 7$
$CD \leq 6, DC \leq 6, CF \leq 3, FC \leq 3$
$DT \leq 4, GD \leq 7$
$FE \leq 2, EH \leq 5, HE \leq 5$
$FG \leq 5, GF \leq 5, FH \leq 3, HF \leq 3$

Inflows must equal outflows:
At A: $SA + EA + CA = AE + AC + AB$
At B: $AB + CB + DB = BT + BD + BC$
At C: $AC + BC + DC + FC = CA + CB + CD + CF$
At D: $BD + CD + GD = DB + DC + DT$
At E: $SE + AE + FE + HE = EA + EH$
At F: $CF + GF + HF = FE + FC + FG + FH$
At G: $FG = GF + GD$
At H: $SH + EH + FH = HE + HF$
[The inflow to T will automatically equal the outflow from S.]

(6) Workers A-E are to be allocated tasks, so that each worker carries out one task, and each task is carried out by one worker. The table below shows the tasks that each worker is trained to do. The aim is to match up workers to tasks in such a way that as many workers as possible are occupied. Formulate this as a linear programming problem.

	1	2	3	4	5
A		Y	Y		Y
B	Y	Y			
C		Y		Y	Y
D	Y		Y		
E		Y		Y	

Solution

With A2, A3, ... , E4 being binary variables, where $A2 = 1$ means that worker A carries out task 2:

Maximise $P = A2 + A3 + A5 + B1 + B2 + C2 + C4 + C5 + D1 + D3 + E2 + E4$

[ie maximise the number of matchings]

subject to the following constraints:

$A2 + A3 + A5 \leq 1$ [ie at most one of $A2, A3, A5$ can be 1]

$B1 + B2 \leq 1$

$C2 + C4 + C5 \leq 1$

$D1 + D3 \leq 1$

$E2 + E4 \leq 1$

$B1 + D1 \leq 1$ [ie at most one of $B1, D1$ can be 1]

$A2 + B2 + C2 + E2 \leq 1$

$A3 + D3 \leq 1$

$C4 + E4 \leq 1$

$A5 + C5 \leq 1$

(7)(i) Workers A-E are to be allocated tasks, so that each worker carries out one task, and each task is carried out by one worker. The table below shows the time taken to train each worker for each task. The aim is to minimise the time spent on training. Formulate this as a linear programming problem.

	1	2	3	4	5
A	4	3	7	2	6
B	2	5	5	4	5
C	3	6	2	6	7
D	4	3	5	7	3
E	3	5	7	4	4

(ii) If in fact worker A cannot carry out task 1, what modification would be necessary?

Solution

(i) With A1, A2, ... , E5 being binary variables, where $A1 = 1$ means that worker A carries out task 1:

Minimise $\quad P = 4A1 + 3A2 + \cdots + 4E5$

[ie minimise the total time spent on training] subject to the following constraints:

$A1 + A2 + A3 + A4 + A5 = 1$ [just one of $A1, A2, A3, A4, A5$ must be 1]

and similarly for B, C, D & E.

$A1 + B1 + C1 + D1 + E1 = 1$

and similarly for 2,3,4 & 5

(ii) To allow for the fact that worker A cannot carry out task 1, increase the element of the table in row A and column 1 from 4 to a large number, such as 100.

(8) A company has 3 warehouses (A,B & C) producing identical items. These have to be delivered to 4 shops, in such a way as to minimise the total transportation cost. These costs are shown in the table below, together with the number of items available at each warehouse (the 'supply'), and the number of items required by each shop (the 'demand'). The aim is to decide how many items each warehouse should deliver to each shop. Formulate this as a linear programming problem.

	demand:	10	11	8	6
supply:		1	2	3	4
12	A	7	4	5	2
13	B	3	6	4	6
10	C	8	3	4	5

Solution

With A1, A2, ... , C4 being non-negative integers, so that A1 is the number of items transported from warehouse A to shop 1:

Minimise $P = 7A1 + 4A2 + \cdots + 5C4$,

subject to the following constraints:

$$A1 + A2 + A3 + A4 = 12$$
$$B1 + B2 + B3 + B4 = 13$$
$$C1 + C2 + C3 + C4 = 10$$

$$A1 + B1 + C1 = 10$$

$$A2 + B2 + C2 = 11$$

$$A3 + B3 + C3 = 8$$

$$A4 + B4 + C4 = 6$$