Linear Programming (9 pages; 21/11/2019)

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Example 1

Maximise $P = x + y$ (eg total number of cakes made)

subject to $2x + 3y \leq 12$ (1st constraint on cost of ingredients)

$6x + 5y \leq 30$ (2nd constraint on cost of ingredients)

$x \geq 0$, $y \geq 0$

x & y are **control variables**

$P = x + y$ is the **objective function**

inequalities are **(linear) constraints**

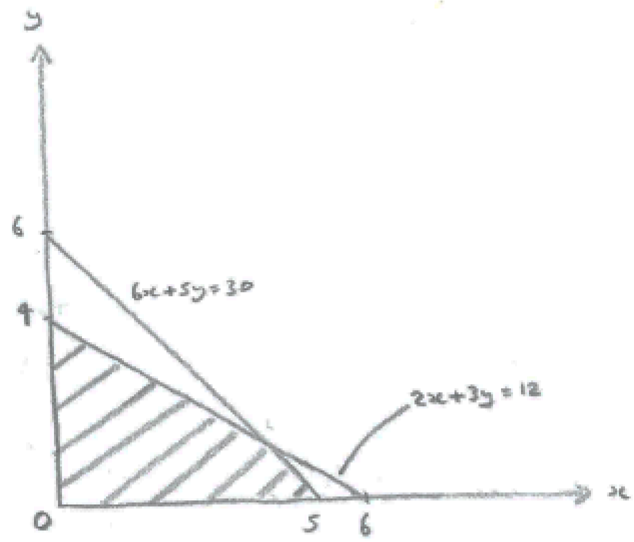


Figure 1

The shaded area in Figure 1 is the **feasible region** (ie where all the constraints are satisfied.

In more complicated cases, the unwanted areas can be shaded, leaving the feasible region as the area with no shading. [This is standard practice for exam purposes.]

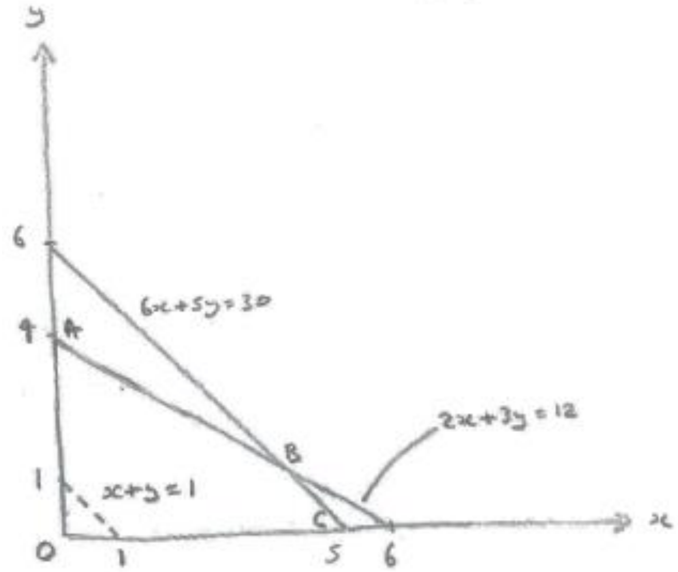


Figure 2

Referring to Figure 2, the line $P = x + y$ will be parallel to $x + y = 1$, and as far away from O as possible.

Unless the line $P = x + y$ has the same gradient as one of the constraint lines, the maximum value of P will occur at one of the vertices of the feasible region. In this example it is B.

If it isn't easy to tell from your sketch, you could simply evaluate the objective function at each vertex.

At B, $6x + 5y = 30$ (1)

& $2x + 3y = 12$ (2)

$6x + 5y = 30$ (1)

$6x + 9y = 36$ (2)

$(2) - (1) \Rightarrow 4y = 6 \Rightarrow y = \frac{3}{2} = 1.5$

$(2) \Rightarrow 2x = 12 - \frac{9}{2} \Rightarrow x = 6 - \frac{9}{4} = \frac{15}{4} = 3.75$

$\Rightarrow P = \frac{21}{4} = 5.25$ at (3.75,1.5)

At A, $P = 4$; at C, $P = 5$; confirming that B is the required vertex.

If only integer values are acceptable (as in this example, where x & y represent numbers of cakes), we can consider points neighbouring (3.75, 1.5), provided they are within the feasible region.

We require $6x + 5y \leq 30$ & $2x + 3y \leq 12$

(3,1): $6x + 5y = 23$ & $2x + 3y = 9$; $P = 4$

(3,2): $6x + 5y = 28$ & $2x + 3y = 12$; $P = 5$

(4,1): $6x + 5y = 29$ & $2x + 3y = 11$; $P = 5$

(4,2): $6x + 5y = 34$ (reject)

Thus the points (3,2) and (4,1) give an equally good solution.

However, there is no guarantee that this will be the optimal solution.

Example 2

Minimise $\mathrm{P} = 2x + y$

subject to $y \geq 3$

$3x + 4y \geq 24$

$y \leq 3x$

$x \geq 0\ ,\ y \geq 0$

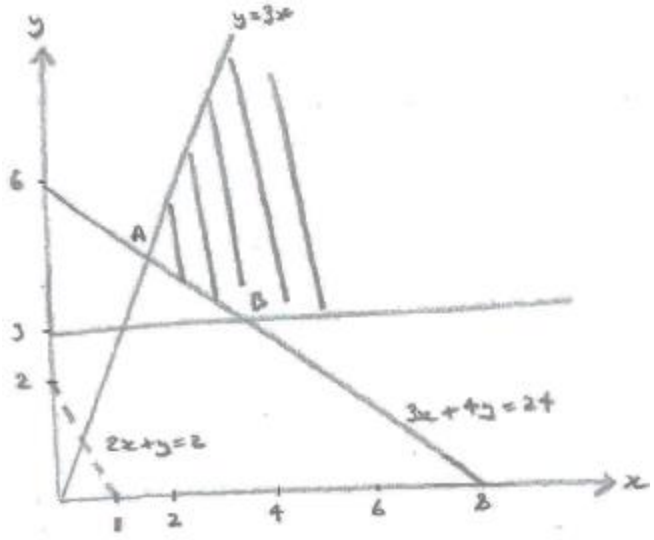


Figure 3

By considering lines parallel to $2x + y = 2$, we can see that P is minimised at A.

$3x + 4y = 24\ \&\ y = 3x \Rightarrow 15x = 24 \Rightarrow x = \frac{8}{5} = 1.6$; thus A is (1.6,4.8), where P = 8

Determining the optimum vertex

Method 1

In many cases, it can be established by eye: having drawn in an example line for the objective function (the dashed line in the above examples), we can imagine this to be moved in such a way that its gradient is kept the same. To maximise the objective function, we move the line as far from the Origin as possible, whilst remaining in the feasible region. To minimise the objective function, we keep the line as near to the Origin as possible, whilst remaining in the feasible region.

Method 2

Bearing in mind that method 1 relies on an accurate drawing, it may be unclear which vertex is the optimum one, if the gradient of a constraint line is similar to that of the objective function. In this case, the objective function can be compared for each of the vertices that could possibly be the optimum one.

Method 3

The gradients of the constraint lines can be compared with that of the (dashed) objective line.

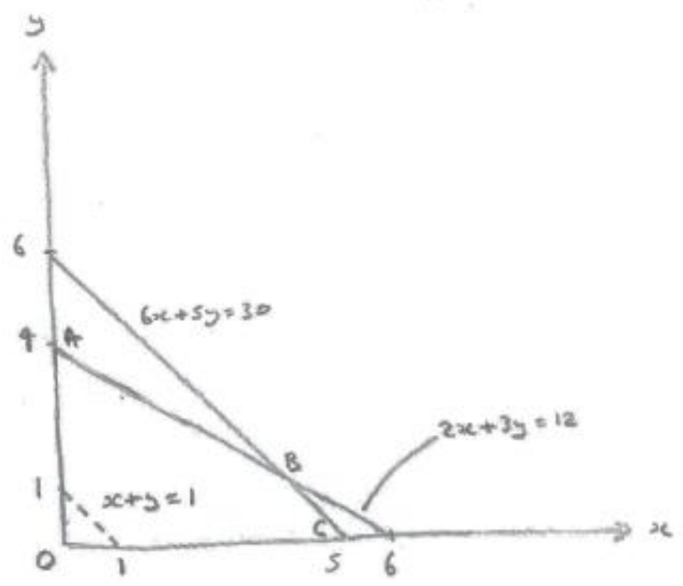
In the case of Example 1 (the graph of which is reproduced below), we can reason as follows:

The gradients are:

objective line ($x + y = 1$): -1

$6x + 5y = 30$: $-\frac{6}{5}$

$2x + 3y = 12$: $-\frac{2}{3}$



Example 1

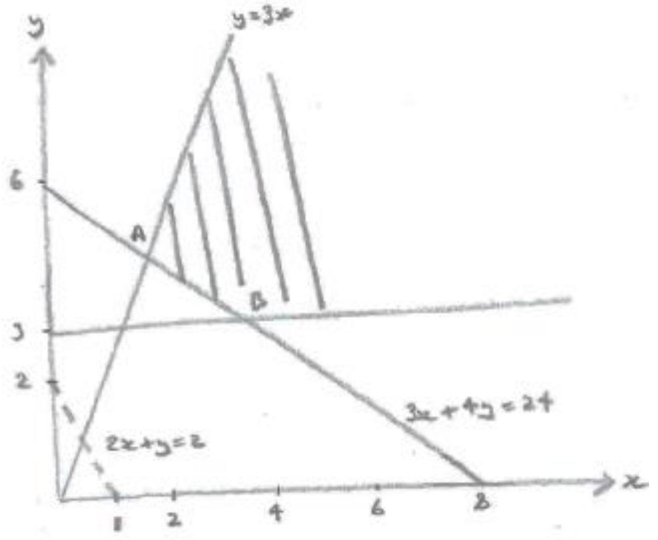
Thus, the objective line is steeper (ie its gradient is more negative) than $2x + 3y = 12$, and the intersection of these lines occurs to the left of the y-axis. This means that, when the moving objective line meets A, there is scope for it to continue moving further away from the Origin (whilst staying in the feasible region for some points on the line), as its intersection point with

$2x + 3y = 12$ moves down $2x + 3y = 12$, towards the x-axis.

Now, comparing the objective line with $6x + 5y = 30$, we see that the latter is steeper (with the intersection point occurring below the x-axis). This means that, in order for the moving objective line to be further away from the Origin, the intersection with

$6x + 5y = 30$ has to be further away from the x-axis. So when B is reached on the line $2x + 3y = 12$ (having moved down from A) we are first of all constrained by the feasible region to move down the line $6x + 5y = 30$, but by the preceding argument we cannot move any further down this line, and so B is the optimum vertex.

In the case of Example 2 (see below), we can imagine the objective line moving in from a long distance from the Origin. The intersection of this objective line with the line $y = 3x$ then occurs for a large value of y, and from the relative slopes of the lines there is scope for the objective line to advance as far as the position where its intersection with $y = 3x$ is at A.



Similarly, as far as the line $y = 3$ is concerned, the objective line can advance as far as the position where its intersection with

$y = 3$ is at B.

We then compare the gradient of the objective line (-2) with that of the constraint line $3x + 4y = 24$ $(-\frac{3}{4})$. As the objective line is the steeper of the two, they will intersect at A after they intersect at B (as the objective line moves in from a long distance from the Origin). Thus A is the last vertex to be reached in this way (or the first if moving from the Origin), and is therefore the optimum vertex.

Note that, for both examples, we consider the intersection of the moving objective line with each constraint line (in turn) and move down the constraint line, away from the point of intersection, until we come to a vertex marking the boundary of the feasible region.

Integer solutions

Method 1

We can consider the integer corners of the square containing the optimum vertex, as described in Example 1. This may not be the best integer solution however.

Method 2

In example 1, the optimum vertex occurs at (3.75, 1.5). As a variant on method 1, we can now consider the cases $x = 3$ and $x = 4$:

When $x = 3$, the objective function (to be maximised) is

$P = 3 + y$, and this means that we need to maximise y.

The constraints become $2(3) + 3y \leq 12$; ie $y \leq 2$,

and $6(3) + 5y \leq 30$; ie $y \leq \frac{12}{5}$

Thus the required integer value of y is 2, and $P = 3 + 2 = 5$.

When $x = 4$, the constraints become $2(4) + 3y \leq 12$; ie $y \leq \frac{4}{3}$,

and $6(4) + 5y \leq 30$; ie $y \leq \frac{6}{5}$

Thus the required integer value of y is 1, and $P = 4 + 1 = 5$.

So the points (3,2) and (4,1) give an equally good solution.

Method 3

For information, there is a more complicated method called the Branch and Bound method, which looks beyond the square in method 1, and will find the best integer solution (this is part of the OCR syllabus).