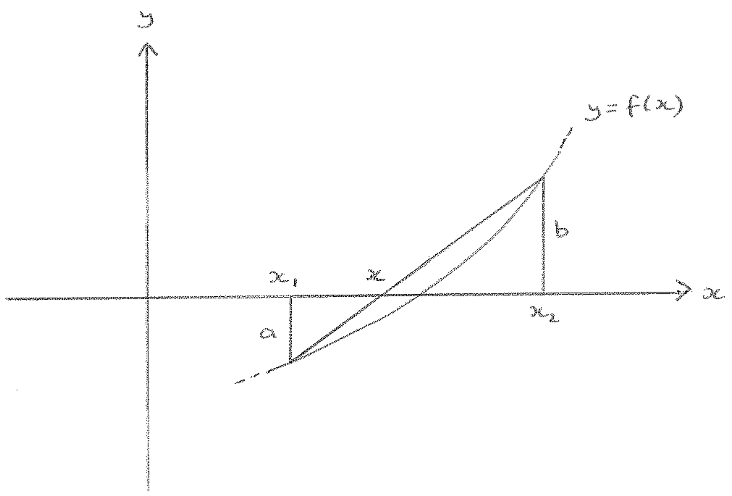
Linear Interpolation (5 pages; 7/2/16)

(1) Theory

Approach A

Example: Suppose that the solution of f(x) = 0 is known to lie between x_1 and x_2, because $f(x_1) = -a$ and $f(x_2) = b$ (where a & b are +ve). We can find an approximate solution using linear interpolation by assuming that f(x) is a straight line between x_1 and x_2 (see below).



By similar triangles, $\frac{b}{a} = \frac{(x_2-x)}{(x-x_1)}$

=> $bx - bx_1 = ax_2 - ax$

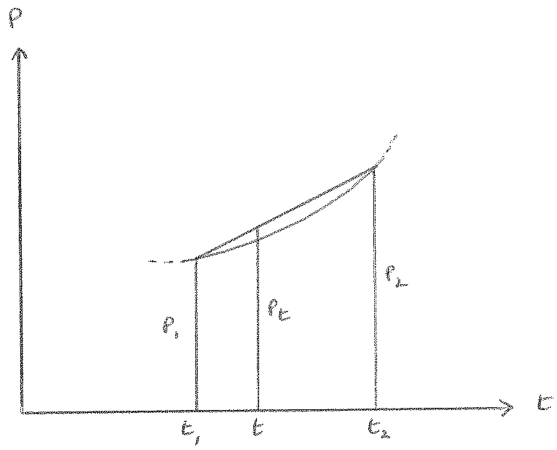
=> $x(a+b) = bx_1 + ax_2$

=> $x = \frac{bx_1+ax_2}{a+b}$

which can be thought of as a weighted average of x_1 and x_2

Approach B

Example: If a population is P_1 at time t_1 and P_2 at time t_2 , linear interpolation can be used to estimate the population P_t at time t, by assuming that the population function is a straight line between P_1 and P_2 (see below).



We want a weighted average of P_1 and P_2.

The two weights are $\frac{(t-t_1)}{(t_2-t_1)}$ and $\frac{(t_2-t)}{(t_2-t_1)}$.

If t is nearer t_1 than t_2 (as in this example), then the larger weight will be applied to P_1, so that:

$$P_t \approx P_1 . \frac{(t_2-t)}{(t_2-t_1)} + P_2 . \frac{(t-t_1)}{(t_2-t_1)}$$

This can also be rearranged as follows:

$$P_t \approx P_1 . \frac{(t_2-t_1)}{(t_2-t_1)} + P_1 . \frac{(t_1-t)}{(t_2-t_1)} + P_2 . \frac{(t-t_1)}{(t_2-t_1)}$$

$$= P_1 + (P_2 - P_1) . \frac{(t-t_1)}{(t_2-t_1)}$$

which can be interpreted as adding on the required proportion of

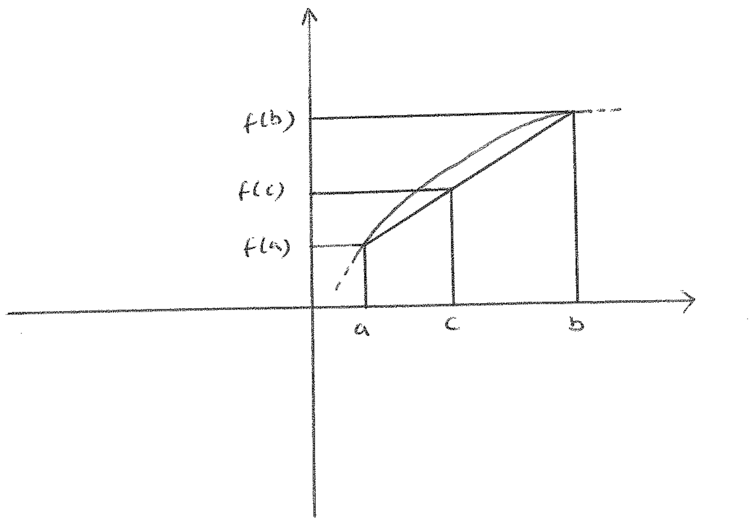
$(P_2 - P_1)$ to P_1.

Approach C

See below. Note that the points (a,f(a)), (b,f(b)) & (c,f(c)) lie on a straight line (where f(c) is the approximation based on linear interpolation).

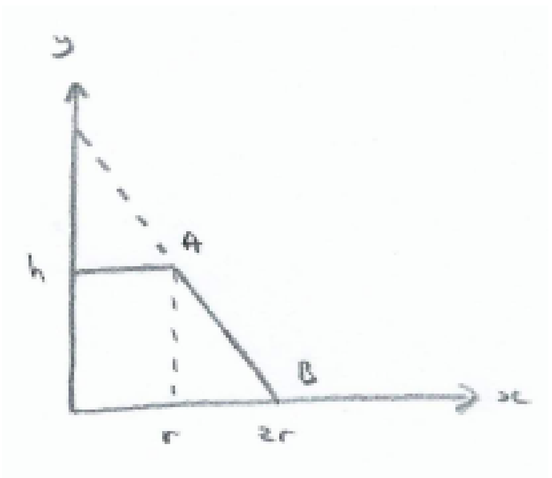
Then f(c) = f(a) + m(c-a) , where m is the gradient of the line

Hence f(c) = f(a) + $\frac{f(b)-f(a)}{b-a}$ (c-a)



(2) Straight Line Equation (involving linear interpolation)

Task: To find the equation of the sloping side of the trapezium (AB), by as many methods as possible (in the form $y = mx + c$).



Method 1a

Coordinates of A and B are (r, h) & $(2r, 0)$.

Hence equation is $\frac{y-0}{x-2r} = \frac{h-0}{r-2r} \Rightarrow y = -\frac{h}{r}(x - 2r) = -\frac{h}{r}x + 2h$

Method 1b

Or $\frac{y-h}{x-r} = \frac{h-0}{r-2r} \Rightarrow y = -\frac{h}{r}(x - r) + h = -\frac{h}{r}x + 2h$

Method 2

gradient is $-\frac{h}{2r}$ and y-intercept is 2h (by similar triangles)

so $y = -\frac{h}{r}x + 2h$

Method 3a

The x-coordinate is r at A (when $y = h$) and 2r at B (when $y = 0$). By linear interpolation, at the general point (x, y) (but easier to consider a point between A and B):

$x = \frac{y}{h}(r) + \frac{h-y}{h}(2r)$

$\Rightarrow xh = -ry + 2hr \Rightarrow y = -\frac{h}{r}x + 2h$

Method 3b

The y-coordinate is h at A (when $x = r$) and 0 at B (when $x = 2r$). By interpolation, at the general point (x, y):

$y = \frac{2r-x}{r}(h) + \frac{x-r}{r}(0) = -\frac{h}{r}x + 2h$

Method 4a

Also by interpolation,

$x = r + \frac{h-y}{h}(r) \Rightarrow hx = hr + (h-y)r \Rightarrow -yr = hx - 2hr$

$\Rightarrow y = -\frac{h}{r}x + 2h$

Method 4b

Or $x = 2r - \frac{y}{h}(r) \Rightarrow hx = 2hr - yr \Rightarrow y = -\frac{h}{r}x + 2h$

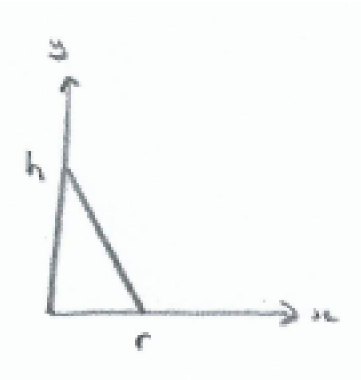
Method 4c

$y = h - \frac{x-r}{r}(h) = -\frac{h}{r}x + 2h$

[Note: $y = 0 + \frac{2r-x}{r}(h) = -\frac{h}{r}x + 2h$ is effectively the same as Method 3b]

Method 5

The line in the diagram below has equation $y = h - \frac{h}{r}x$ (having y-intercept of h and gradient $-\frac{h}{r}$)



Our line can be obtained by translating the above line by r to the right, which is achieved by replacing x with $x - r$.

Thus the new equation is $y = h - \frac{h}{r}(x - r) = -\frac{h}{r}x + 2h$