Lagrange's method (2 pages; 27/3/20)

(1) This is one of two standard methods for approximating functions - the other being Newton's Forward Difference method.

It can be used where the x values are not evenly spaced.

(2) To find the equation of the line that passes through the points (a,A) and (b,B): $\frac{y-A}{x-a} = \frac{B-A}{b-a}$

$\Rightarrow y = \frac{(B-A)(x-a)}{b-a} + A = \frac{B(x-a)}{b-a} + \frac{A(x-a)+A(a-b)}{a-b}$

$= \frac{B(x-a)}{b-a} + \frac{A(x-b)}{a-b}$

(3) Also, it can be seen that

$f(x) = \frac{A(x-b)(x-c)}{(a-b)(a-c)} + \frac{B(x-a)(x-c)}{(b-a)(b-c)} + \frac{C(x-a)(x-b)}{(c-a)(c-b)}$

is a quadratic function that passes through (a,A), (b,B) & (c,C).

The formula can be extended to more than 3 points

(4) **Example**: If $f(x)$ passes through the points (2,5), (3,8) & (4,13),

(i) Find the quadratic function obtained by Lagrange's method

(ii) Estimate $f(2.5)$

Solution

(i) $f(x) = 5\frac{(x-3)(x-4)}{(2-3)(2-4)} + 8\frac{(x-2)(x-4)}{(3-2)(3-4)} + 13\frac{(x-2)(x-3)}{(4-2)(4-3)}$

$= \frac{5}{2}(x-3)(x-4) - 8(x-2)(x-4) + \frac{13}{2}(x-2)(x-3)$

$= x^2\left(\frac{5}{2} - 8 + \frac{13}{2}\right) + x\left(-\frac{35}{2} + 48 - \frac{65}{2}\right) + 30 - 64 + 39$

$= x^2 - 2x + 5$

(ii) $f(2.5) = \frac{5(-0.5)(-1.5)}{(-1)(-2)} + \frac{8(0.5)(-1.5)}{(1)(-1)} + \frac{13(0.5)(-0.5)}{(2)(1)}$

$= \ \frac{15}{8} + 6 - \frac{13}{8} = 6.25$

(5) Note that the same result is obtained as for Newton's forward difference interpolation method, but more work is involved.

(6) Exercise

The distance travelled by a particle, as a function of time, is modelled by the quadratic function $s(t)$. Given the following approximate data, use Lagrange's method to estimate $s(10)$.

t	1	4	8
$s(t)$	2	9	42

Solution

$f(x) = 2\frac{(x-4)(x-8)}{(1-4)(1-8)} + 9\frac{(x-1)(x-8)}{(4-1)(4-8)} + 42\frac{(x-1)(x-4)}{(8-1)(8-4)}$

Then $f(10) = 2\frac{(6)(2)}{21} + 9\frac{(9)(2)}{(-12)} + 42\frac{(9)(6)}{28} = 68.643 = 69$ (2sf)