Linear Programming & Simplex – Exercises (with Sol'ns)

(36 pages; 1/3/26)

Linear Programming

(1) A company makes sofas and upholstered chairs. Each sofa requires $1m^3$ of material and 14 hours of labour to make, and sells for a profit of £200. Each chair requires $0.2m^3$ of material and 4 hours of labour to make, and sells for a profit of £30. Given that $50m^3$ of material and 840 hours of labour are available, use Linear Programming to find the number of sofas and chairs that are required, in order to optimise profit, commenting on your answer.

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Solution

The objective function to be maximised is:

$P = 200s + 30c$, where s and c are the numbers of sofas and chairs made;

subject to the following constraints:

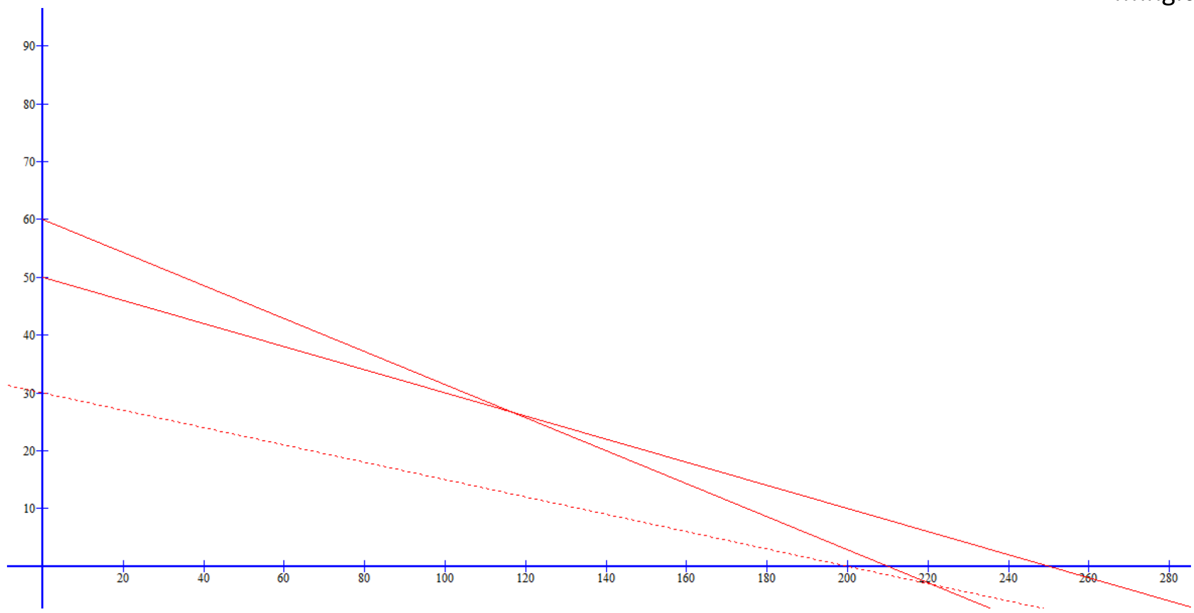
$s + 0.2c \leq 50$ or $5s + c \leq 250$

$14s + 4c \leq 840$ or $7s + 2c \leq 420$

$t, c \geq 0$ (integers)

[For exam purposes, you may wish to use a letter other than s, to avoid confusion with the number 5.]

The diagram (with s against c) shows the constraint lines and feasible region, as well as the (dotted) line $6000 = 200s + 30c$, which is parallel to the objective function.



[As the gradient of the line representing the objective function is close to that of one of the constraint lines, it may not be clear which of the vertices of the feasible region maximises P (ie which vertex is the last to be passed through, as the objective function moves away from the Origin). Instead we can determine the value of P at the two likely vertices.]

At (0,50) [A], $P = 200(50) = 10000$

At the intersection of the constraint lines [B],

$5s + c = 250$ and $7s + 2c = 420$,

so that $10s + 2c = 500$ and $3s = 80; s = \frac{80}{3}$
and $c = 250 - 5\left(\frac{80}{3}\right) = \frac{350}{3}$

Then $P = 200\left(\frac{80}{3}\right) + 30\left(\frac{350}{3}\right) = \frac{26500}{3} = 8833$

So P is maximised at A, where $s = 50$ and $c = 0$, so that 50 sofas and no chairs should be made.

[Had the optimal solution occurred at B, an integer solution would need to be found.]

However, this may not be a sensible solution for the company, for the following reasons:

- customers may wish to buy chairs to go with their sofa

- customers may be disappointed at the lack of chairs (when they are not buying a sofa)

- the company might prefer to maintain its capacity to make chairs

(2) The following Linear Programming problem is to be solved:

Minimise $P = 3x + 2y$,

subject to $5x + 3y \geq 20$

$$y \leq 3x$$
$$x \geq 0, y \geq 1$$

(i) Obtain a solution using a graphical approach. Assume that non-integer solutions are acceptable.

(ii) Obtain a good integer solution.

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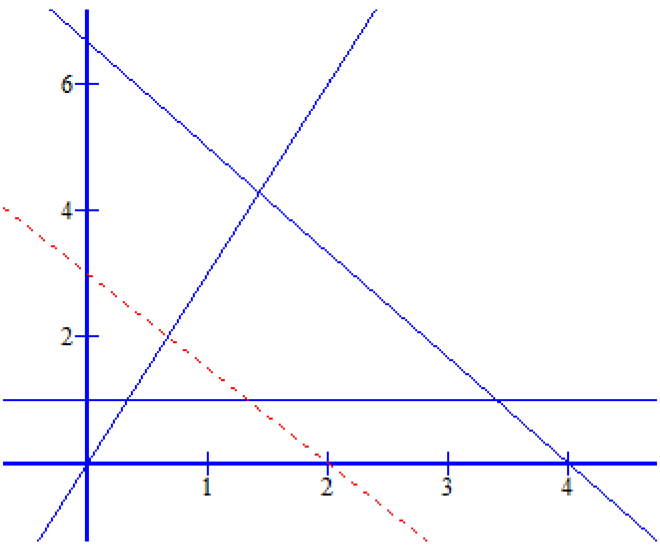
Solution

(i) The diagram shows the constraint lines, as well as the (dotted) line $3x + 2y = 6$, which is parallel to the objective function.



As the line representing the objective function moves away from the Origin, it first enters the feasible region at the intersection of $5x + 3y = 20$ and $y = 1$; ie at the vertex $(3\frac{2}{5}, 1)$, when

$P = 3(3.4) + 2(1) = 12.2$

(ii) To establish an integer solution, consider first of all $x = 3$:

Then $P = 3x + 2y$, and $5x + 3y \geq 20$, $y \leq 3x$ & $y \geq 1$

So $3y \geq 5$; ie $y \geq \frac{5}{3}$, and $y \leq 9$, and $y \geq 1$,

so that $y = 2$ minimises $P = 3x + 2y$, with $P = 13$.

Then consider $x = 4$, which gives $3y \geq 0$ and $y \leq 12$ & $y \geq 1$

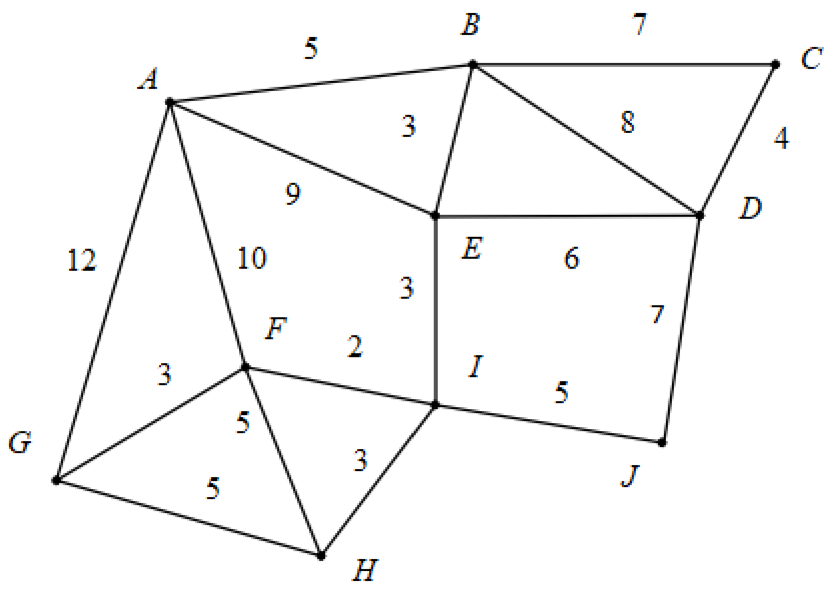
so that $y = 1$ minimises $P = 3x + 2y$, with $P = 14$.

Thus the required integer solution is $(3,2)$, with $P = 13$.

[Strictly speaking, this is just a good integer solution, as it might be the case that the best integer solution lies elsewhere in the feasible region.]

Formulating as LP problem

(1) It is required to find the shortest distance between A and J in the network below. Formulate this as a linear programming problem.



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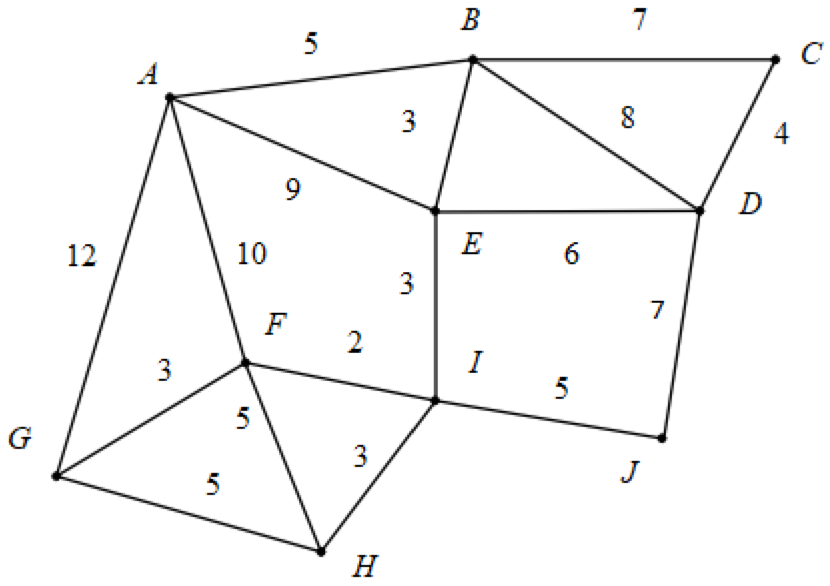


Solution

With AB, AE etc being binary variables, where $AB = 1$ means that the arc AB is travelled along:

Minimise $P = 5AB + 9AE + 10AF + 12AG + 7BC + 7CB + 8BD + 8DB + 3BE + 3EB + 4CD + 4DC + 6DE + 6ED + 7DJ + 3EI + 3IE + 3FG + 3GF + 2FI + 2IF + 5FH + 5HF + 5GH + 5HG + 3HI + 3IH + 5IJ$

[Arcs not involving A or J can be travelled along in either direction, and so are duplicated.]

$AB + AE + AF + AG = 1$ [the path has to pass along just one of the arcs leading from A]

$DJ + IJ = 1$ [the path has to pass along just one of the arcs leading to J]

$AB + EB + DB + CB = BE + BD + BC$

[if we enter B, then we must leave it - each side will total either 0 or 1]

$BC + DC = CB + CD$ [similarly for C]

$BD + CD + ED = DB + DC + DE + DJ$ [D]

$AE + BE + DE + IE = EB + ED + EI$ [E]

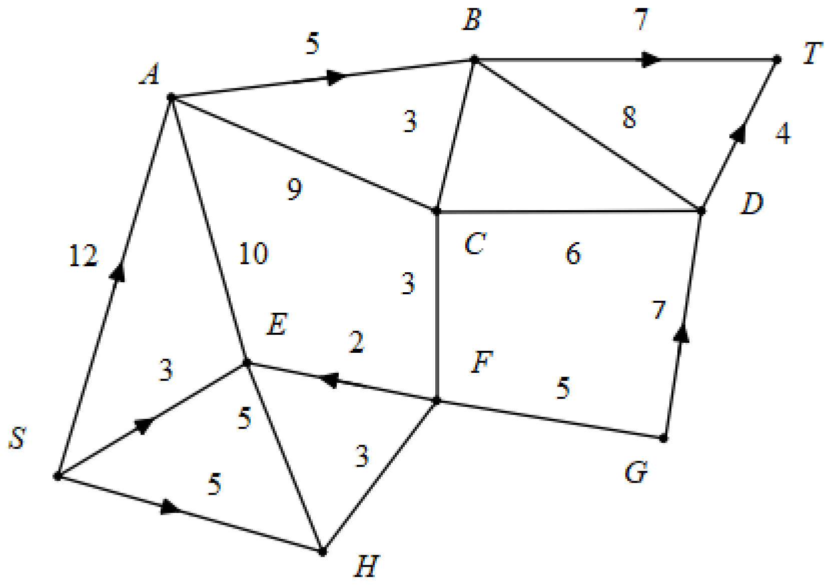
$AF + GF + HF + IF = FG + FH + FI$ [F]

$AG + FG + HG = \mathrm{G}F + GH$ [G]

$FH + GH + IH = HF + HG + HI$ [H]

$EI + FI + HI = IE + IF + IH + IJ$ [I]

(2) The network below shows the maximum capacity for each arc of a network. It is required to maximise the flow across the network, from S to T. Formulate this as a linear programming problem.



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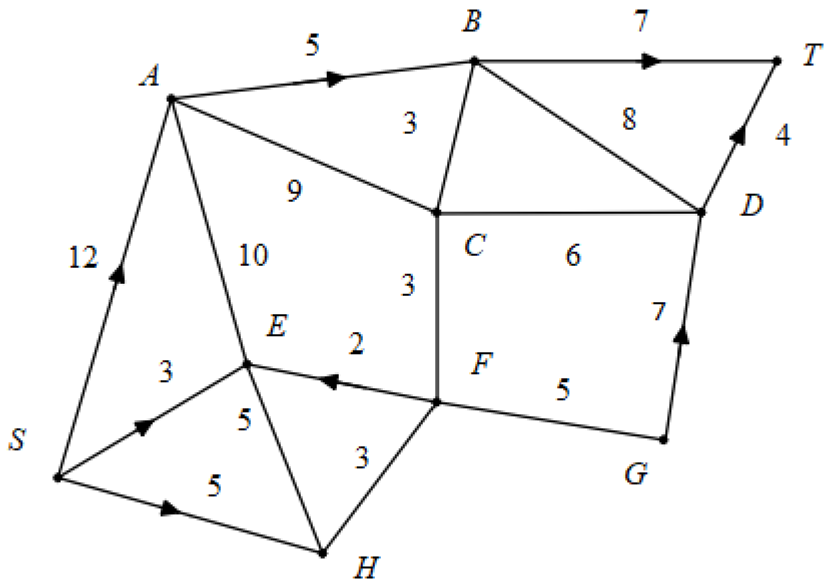

Solution

With SA, AB etc being non-negative integers, representing the flows along the arcs:

Maximise $P = \text{SA} + \text{SE} + \text{SH}$,

subject to the following constraints:

$SA \leq 12, SE \leq 3, SH \leq 5$
$AB \leq 5, AC \leq 9, CA \leq 9, AE \leq 10, EA \leq 10$
$BC \leq 3, CB \leq 3, BD \leq 8, DB \leq 8, BT \leq 7$
$CD \leq 6, DC \leq 6, CF \leq 3, FC \leq 3$
$DT \leq 4, GD \leq 7$
$FE \leq 2, EH \leq 5, HE \leq 5$
$FG \leq 5, GF \leq 5, FH \leq 3, HF \leq 3$

Inflows must equal outflows:
At A: $SA + EA + CA = AE + AC + AB$
At B: $AB + CB + DB = BT + BD + BC$
At C: $AC + BC + DC + FC = CA + CB + CD + CF$
At D: $BD + CD + GD = DB + DC + DT$
At E: $SE + AE + FE + HE = EA + EH$
At F: $CF + GF + HF = FE + FC + FG + FH$
At G: $FG = GF + GD$
At H: $SH + EH + FH = HE + HF$

[The inflow to T will automatically equal the outflow from S.]

(3) Workers A-E are to be allocated tasks, so that each worker carries out one task, and each task is carried out by one worker. The table below shows the tasks that each worker is trained to do. The aim is to match up workers to tasks in such a way that as many workers as possible are occupied. Formulate this as a linear programming problem.

	1	2	3	4	5
A		Y	Y		Y
B	Y	Y			
C		Y		Y	Y
D	Y		Y		
E		Y		Y	

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	1	2	3	4	5
A		Y	Y		Y
B	Y	Y			
C		Y		Y	Y
D	Y		Y		
E		Y		Y	

Solution

With A2, A3, ... , E4 being binary variables, where $A2 = 1$ means that worker A carries out task 2:

Maximise $P = A2 + A3 + A5 + B1 + B2 + C2 + C4 + C5 + D1 + D3 + E2 + E4$

[ie maximise the number of matchings]

subject to the following constraints:

$A2 + A3 + A5 \leq 1$ [ie at most one of $A2, A3, A5$ can be 1]

$B1 + B2 \leq 1$

$C2 + C4 + C5 \leq 1$

$D1 + D3 \leq 1$

$E2 + E4 \leq 1$

$B1 + D1 \leq 1$ [ie at most one of $B1, D1$ can be 1]

$A2 + B2 + C2 + E2 \leq 1$

$A3 + D3 \leq 1$

$C4 + E4 \leq 1$

$A5 + C5 \leq 1$

(4)(i) Workers A-E are to be allocated tasks, so that each worker carries out one task, and each task is carried out by one worker. The table below shows the time taken to train each worker for each task. The aim is to minimise the time spent on training. Formulate this as a linear programming problem.

	1	2	3	4	5
A	4	3	7	2	6
B	2	5	5	4	5
C	3	6	2	6	7
D	4	3	5	7	3
E	3	5	7	4	4

(ii) If in fact worker A cannot carry out task 1, what modification would be necessary?

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C	3	6	2	6	7
D	4	3	5	7	3
E	3	5	7	4	4

(ii) If in fact worker A cannot carry out task 1, what modification would be necessary?

Solution

(i) With A1, A2, ... , E5 being binary variables, where $A1 = 1$ means that worker A carries out task 1:

Minimise $\quad P = 4A1 + 3A2 + \cdots + 4E5$

[ie minimise the total time spent on training]

subject to the following constraints:

$A1 + A2 + A3 + A4 + A5 = 1$ [just one of $A1, A2, A3, A4, A5$ must be 1]

and similarly for B, C, D & E

$A1 + B1 + C1 + D1 + E1 = 1$

and similarly for 2,3,4 & 5

(ii) To allow for the fact that worker A cannot carry out task 1, increase the element of the table in row A and column 1 from 4 to a large number, such as 100.

(5) A company has 3 warehouses (A,B & C) producing identical items. These have to be delivered to 4 shops, in such a way as to minimise the total transportation cost. These costs are shown in the table below, together with the number of items available at each warehouse (the 'supply'), and the number of items required by each shop (the 'demand'). The aim is to decide how many items each warehouse should deliver to each shop. Formulate this as a linear programming problem.

	demand:	10	11	8	6
supply:		1	2	3	4
12	A	7	4	5	2
13	B	3	6	4	6
10	C	8	3	4	5

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12	A	7	4	5	2
13	B	3	6	4	6
10	C	8	3	4	5

Solution

With A1, A2, ... , C4 being non-negative integers, so that A1 is the number of items transported from warehouse A to shop 1:

Minimise $P = 7A1 + 4A2 + \cdots + 5C4$,

subject to the following constraints:

$A1 + A2 + A3 + A4 = 12$

$B1 + B2 + B3 + B4 = 13$

$C1 + C2 + C3 + C4 = 10$

$A1 + B1 + C1 = 10$

$A2 + B2 + C2 = 11$

$$A3 + B3 + C3 = 8$$

$$A4 + B4 + C4 = 6$$

Simplex method

(1) The following Linear Programming problem is to be solved:

Minimise $P = 3x + 2y$,

subject to $5x + 3y \geq 20$
$y \leq 3x$
$x \geq 0, y \geq 1$

Apply the Big M Simplex method, up to the point where the 1st pivot has been completed, and the 2nd is about to be carried out.

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Solution

We wish to maximise $-P = -3x - 2y$

The constraint equations are:

$5x + 3y - s_1 + a_1 = 20$

$y - 3x + s_2 = 0$

$y - s_3 + a_2 = 1$

$x, s_1, s_2, s_3 \geq 0$

Let $P' = -P - M(a_1 + a_2)$

$= -3x - 2y - M(20 - 5x - 3y + s_1) - M(1 - y + s_3)$

so that $P' + x(3 - 5M) + y(2 - 4M) + Ms_1 + Ms_3 = -21M$

The initial Simplex tableau is

basic variable	x	y	s_1	s_2	s_3	a_1	a_2	value
a_1	5	3	-1	0	0	1	0	20
s_2	-3	1	0	1	0	0	0	0
a_2	0	1	0	0	-1	0	1	1
P'	$3-5M$	$2-4M$	M	0	M	0	0	$-21M$

Take x as the pivot column [as $3-5M < 2-4M$]

basic variable	x	y	s_1	s_2	s_3	a_1	a_2	value	ratio
a_1	5	3	-1	0	0	1	0	20	4
s_2	-3	1	0	1	0	0	0	0	n/a
a_2	0	1	0	0	-1	0	1	1	n/a
P'	$3-5M$	$2-4M$	M	0	M	0	0	$-21M$	n/a

Applying the ratio test, the pivot row is found to be the 1st one.

Making the coeff. of x equal to 1:

basic variable	x	y	s_1	s_2	s_3	a_1	a_2	value	row
a_1	1	0.6	-0.2	0	0	0.2	0	4	1
s_2	-3	1	0	1	0	0	0	0	2
a_2	0	1	0	0	-1	0	1	1	3
P'	$3 - 5M$	$2 - 4M$	M	0	M	0	0	$-21M$	4

Remove x from rows 2, 3 & 4:

$R2' = R2 + 3(R1'), R3' = R3, R4' = R4 + (5M - 3)(R1')$

basic variable	x	y	s_1	s_2	s_3	a_1	a_2	value	row
x	1	0.6	-0.2	0	0	0.2	0	4	$1'$
s_2	0	2.8	-0.6	1	0	0.6	0	12	$2'$
a_2	0	1	0	0	-1	0	1	1	$3'$
P'	0	$0.2 - M$	0.6	0	M	$M - 0.6$	0	$-M - 12$	$4'$

(2) Minimise $-3x + 2y + z$, subject to the following constraints:

$x + y - 4z \leq 4$

$-x + 3y + 2z \geq -2$

$x \geq 0, y \geq 0, z \geq 0$

Use the ordinary Simplex method to solve this problem.

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Solution

Step 1: Rewrite the problem as

Maximise $P = 3x - 2y - z$,

subject to $x + y - 4z \leq 4$ and $x - 3y - 2z \leq 2$

Step 2: Create equations with slack variables [it is possible to skip this step, and go straight to the Simplex tableau]:

$P - 3x + 2y + z = 0$ (1)

$x + y - 4z + s_1 = 4$ (2)

$x - 3y - 2z + s_2 = 2$ (3)

Step 3: Represent the equations in a Simplex tableau:

P	x	y	z	s_1	s_2	value	row
1	-3	2	1	0	0	0	(1)
0	1	1	-4	1	0	4	(2)
0	(1)	-3	-2	0	1	2	(3)

Step 4: Choose x as the pivot column (as it has the largest negative coefficient in the objective row), and perform the ratio test to establish the pivot row.

As $\frac{2}{1} < \frac{4}{1}$, row 3 is the pivot row (indicated in the table above by the brackets - or circling if handwritten).

Step 5: Eliminate x from rows 1 and 2

As the coefficient of x for row 3 is already 1, no adjustment is needed for that row.

P	x	y	z	s_1	s_2	value	row
1	0	-7	-5	0	3	6	(4)=(1)+3(6)
0	0	(4)	-2	1	-1	2	(5)=(2)-(6)
0	1	-3	-2	0	1	2	(6)=(3)

Step 6: y now has the largest negative coefficient in the objective row, and as the coefficient of y in row 6 is negative, we can take row 5 as the pivot row.

Step 7: Eliminate y from rows 4 and 6

As the coefficient of y for row 5 is 4, we need to divide that row by 4 first.

P	x	y	z	s_1	s_2	value	row
1	0	0	$-8\frac{1}{2}$	$1\frac{3}{4}$	$1\frac{1}{4}$	$9\frac{1}{2}$	(7)=(4)+7(8)
0	0	1	$-\frac{1}{2}$	$\frac{1}{4}$	$-\frac{1}{4}$	$\frac{1}{2}$	(8)=(5)÷ 4
0	1	0	$-3\frac{1}{2}$	$\frac{3}{4}$	$\frac{1}{4}$	$3\frac{1}{2}$	(9)=(6)+3(8)

Step 8: Although z has a negative coefficient in the objective row, the other coefficients of z are negative, and so no further progress can be made.

Hence the solution is: $x = 3\frac{1}{2}$, $y = \frac{1}{2}$, $z = 0, s_1 = 0, s_2 = 0, P = 9\frac{1}{2}$,

and hence the minimised value of $-3x + 2y + z$ is $-9\frac{1}{2}$

[Check: $x + y - 4z = 4 \leq 4$ and $-x + 3y + 2z = -2 \geq -2$]

(3) Maximise $5x - 2y + 4z$, subject to the following constraints:

$2x + y - z \leq 6$

$x - y + 2z \geq 5$

$3x + y - 7z \geq 4$

$x \geq 0, y \geq 0, z \geq 0$

(i) Apply the 1st stage of the 2 Stage Simplex method, as far as establishing the pivot row for the 2nd time.

(ii) Instead, apply the Big M (Simplex) method, as far as establishing the pivot row for the 2nd time.

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Solution

(i) Step 1: Create equations with either slack variables, or surplus and artifical variables, as appropriate.

$P - 5x + 2y - 4z = 0$ (1)

$2x + y - z + s_1 = 6$ (2)

$x - y + 2z - s_2 + a_1 = 5$ (3)

$3x + y - 7z - s_3 + a_2 = 4$ (4)

Step 2: Let $A = a_1 + a_2 = (5 - x + y - 2z + s_2) + (4 - 3x - y + 7z + s_3)$

$= 9 - 4x + 5z + s_2 + s_3$

The 1st stage of the method is to minimise A.

Step 3: Represent the equations in a Simplex tableau:

A	P	x	y	z	s_1	s_2	s_3	a_1	a_2	value	row
1	0	4	0	-5	0	-1	-1	0	0	9	(1)
0	1	-5	2	-4	0	0	0	0	0	0	(2)
0	0	2	1	-1	1	0	0	0	0	6	(3)
0	0	1	-1	2	0	-1	0	1	0	5	(4)
0	0	(3)	1	-7	0	0	-1	0	1	4	(5)

Step 4: As we are minimising A, we look for large positive coefficients of variables in the 1st row (so that when the variable is maximised, it will reduce A as much as possible). Here there is no choice, and we take x as the pivot column, and perform the ratio test to establish the pivot row (ignoring any rows with negative coefficients of x).

row 3: $\frac{6}{2} = 3$; row 4: $\frac{5}{1} = 5$; row 5: $\frac{4}{3}$; so the pivot row is row 5 (indicated in the table above by the brackets - or circling if handwritten)

[Note: It is possible (though less usual) to maximise $-A$ instead.]

Step 5: Eliminate x from rows 1-4

A	P	x	y	z	s_1	s_2	s_3	a_1	a_2	value	row
1	0	0	$-\frac{4}{3}$	$\frac{13}{3}$	0	-1	$\frac{1}{3}$	0	$-\frac{4}{3}$	$\frac{11}{3}$	(6)=(1)−4(10)
0	1	0	$\frac{11}{3}$	$-\frac{47}{3}$	0	0	$-\frac{5}{3}$	0	$\frac{5}{3}$	$\frac{20}{3}$	(7)=(2)+5(10)

0	0	0	$\frac{1}{3}$	$\frac{11}{3}$	1	0	$\frac{2}{3}$	0	$-\frac{2}{3}$	$\frac{10}{3}$	(8)=(3) −2(10)
0	0	0	$-\frac{4}{3}$	$(\frac{13}{3})$	0	-1	$\frac{1}{3}$	1	$-\frac{1}{3}$	$\frac{11}{3}$	(9)=(4) −(10)
0	0	1	$\frac{1}{3}$	$-\frac{7}{3}$	0	0	$-\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{4}{3}$	(10)=(5)÷ 3

Step 6: As A hasn't yet been reduced to zero, we look for large positive coefficients of variables in the 1st row again, and so take z as the pivot column. Rows 7 and 10 can be ignored, when establishing the pivot row, due to their negative coefficients of z.

row 8: $\frac{(\frac{10}{3})}{(\frac{11}{3})} = \frac{10}{11}$; row 9: $\frac{(\frac{11}{3})}{(\frac{13}{3})} = \frac{11}{13} < \frac{10}{11}$, so the pivot row is row 9

(ii) Step 1: (As for the 2 Stage method), create equations with either slack variables, or surplus and artifical variables, as appropriate

$P - 5x + 2y - 4z = 0$ (1)

$2x + y - z + s_1 = 6$ (2)

$x - y + 2z - s_2 + a_1 = 5$ (3)

$3x + y - 7z - s_3 + a_2 = 4$ (4)

Step 2: Modify the objective to maximising $P' = 5x - 2y + 4z - M(a_1 + a_2)$

$= 5x - 2y + 4z - M[(5 - x + y - 2z + s_2) + (4 - 3x - y + 7z + s_3)]$

$= (5+4M)x - 2y + (4-5M)z - Ms_2 - Ms_3 - 9M$

(where M is a large number)

Step 3: Represent the equations in a Simplex tableau:

P'	x	y	z	s_1	s_2	s_3	a_1	a_2	value	row
1	$-5-4M$	2	$-4+5M$	0	M	M	0	0	$-9M$	(1)
0	2	1	-1	1	0	0	0	0	6	(2)
0	1	-1	2	0	-1	0	1	0	5	(3)
0	(3)	1	-7	0	0	-1	0	1	4	(4)

Step 4: As we are maximising P', we look for large negative coefficients of variables in the 1st row (so that when the variable is maximised, it will increase P' as much as possible). Here we take x as the pivot column, and perform the ratio test to establish the pivot row.

row 2: $\frac{6}{2} = 3$; row 3: $\frac{5}{1} = 5$; row 4: $\frac{4}{3}$; so the pivot row is row 4 (indicated in the table above by the brackets - or circling if handwritten)

Step 5: Eliminate x from rows 1-3

P'	x	y	z	s_1	s_2	s_3	a_1	a_2	value	row
1	0	$\frac{4M+11}{3}$	$\frac{-13M-47}{3}$	0	M	$\frac{-M-5}{3}$	0	$\frac{5+4M}{3}$	$\frac{-11M+20}{3}$	(5)=(1)+ (5+4M)(8)
0	0	$\frac{1}{3}$	$\frac{11}{3}$	1	0	$\frac{2}{3}$	0	$-\frac{2}{3}$	$\frac{10}{3}$	(6)=(2) -2(8)

0	0	$-\frac{4}{3}$	$(\frac{20}{3})$	0	-1	$\frac{1}{3}$	1	$-\frac{1}{3}$	$\frac{11}{3}$	(7)= (3) − (8)
0	1	$\frac{1}{3}$	$-\frac{7}{3}$	0	0	$-\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{4}{3}$	(8)=(4)÷ 3

Step 6: As the value of P' still involves M, we look for large negative coefficients of variables in the 1st row again, and so take z as the pivot column. Row 8 can be ignored, when establishing the pivot row, due to its negative coefficient of z.

row 6: $\frac{(\frac{10}{3})}{(\frac{11}{3})} = \frac{10}{11}$; row 7: $\frac{(\frac{11}{3})}{(\frac{20}{3})} = \frac{11}{20} < \frac{10}{11}$, so the pivot row is row 7