Kinematics Exercises (Solutions) (3 pages; 26/1/20)

(1***) Given that $v(x) = 10e^{-x}$ and that $x = 0$ when $t = 0$, find:

(i) the acceleration as a function of x

(ii) x as a function of t

(iii) v as a function of t

(iv) the acceleration as a function of t

Solution

(i) $a(x) = \frac{dv}{dt} = \frac{dv}{dx} . \frac{dx}{dt} = v\frac{dv}{dx} = 10e^{-x}(-10e^{-x}) = -100e^{-2x}$

[The relation $a(x) = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$ is sometimes more convenient to use.]

(ii) $\frac{dx}{dt} = 10e^{-x}$

$\Rightarrow \int e^x dx = 10\int dt$

$\Rightarrow e^x = 10t + C$

$x = 0$ when $t = 0 \Rightarrow C = 1$

$\Rightarrow x(t) = \ln(10t + 1)$

(iii) From (ii), $v(t) = 10e^{-\ln(10t+1)}$

$\Rightarrow v(t) = 10(10t + 1)^{-1}$

(iv) From (i) & (ii), $a(t) = -100e^{-2\ln(10t+1)}$

$\Rightarrow a(t) = -100(10t+1)^{-2}$

Check:

$\frac{d}{dt}x(t) = \frac{1}{10t+1}(10) = 10(10t+1)^{-1} = v(t)$

$\frac{d}{dt}v(t) = 10(-1)(10t+1)^{-2}(10)$

$= -100(10t+1)^{-2} = a(t)$

(2***) Given that the acceleration of a particle, $a(x) = x + 1$, and that $x = 0$ and $v = 1$ when $t = 0$, find x in terms of t for $x > 0$.

Solution

Using the relation $\int a(x)dx = \frac{1}{2}v^2$ (*),

[$a = \frac{dv}{dt} = \frac{dv}{dx}.\frac{dx}{dt} = v\frac{dv}{dx}$ and $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = v\frac{dv}{dx}$

Also, multiplying (*) by mass and applying limits gives

$\int_{x_1}^{x_2} F(x)dx = \frac{1}{2}m{v_2}^2 - \frac{1}{2}m{v_1}^2$, where $F(x)$ is force;

ie the work -energy principle: work done = increase in kinetic energy]

$\frac{1}{2}v^2 = \int x + 1\, dx = \frac{1}{2}x^2 + x + C$

$x = 0, v = 1 \Rightarrow C = \frac{1}{2}$

$\Rightarrow v^2 = x^2 + 2x + 1 = (x+1)^2$

$\Rightarrow v = x + 1$ ($v > 0$ for $x > 0$, as $a > 0$)

So $\frac{dx}{dt} = x + 1$

and hence $\int \frac{1}{x+1} dx = \int dt$

$\Rightarrow \ln(x+1) = t + C \Rightarrow x + 1 = e^{t+C}$ or $x = Ae^t - 1$

$t = 0,\ x = 0 \Rightarrow A = 1$, so that $x = e^t - 1$

Check: $\mathrm{v} = \frac{dx}{dt} = e^t = x + 1$

and $a = \frac{dv}{dt} = e^t = x + 1$