

Kinematics – Exercises (with Sol'ns) (5 pages; 9/2/26)

(1) Given that the velocity of a particle as a function of its displacement is $v(x) = 10e^{-x}$ and that $x = 0$ when $t = 0$, find:

- (i) the acceleration as a function of x
- (ii) x as a function of t
- (iii) v as a function of t
- (iv) the acceleration as a function of t

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(i) the acceleration as a function of x

(ii) x as a function of t

(iii) v as a function of t

(iv) the acceleration as a function of t

Solution

$$(i) a(x) = \frac{dv}{dt} = \frac{dv}{dx} \cdot \frac{dx}{dt} = v \frac{dv}{dx} = 10e^{-x}(-10e^{-x}) = -100e^{-2x}$$

[The relation $a(x) = \frac{d}{dx} \left(\frac{1}{2} v^2 \right)$ is sometimes more convenient to use.]

$$(ii) \frac{dx}{dt} = 10e^{-x}$$

$$\Rightarrow \int e^x dx = 10 \int dt$$

$$\Rightarrow e^x = 10t + C$$

$$x = 0 \text{ when } t = 0 \Rightarrow C = 1$$

$$\Rightarrow x(t) = \ln(10t + 1)$$

$$(iii) \text{ From (ii), } v(t) = 10e^{-\ln(10t+1)}$$

$$\Rightarrow v(t) = 10(10t + 1)^{-1}$$

$$(iv) \text{ From (i) \& (ii), } a(t) = -100e^{-2\ln(10t+1)}$$

$$\Rightarrow a(t) = -100(10t + 1)^{-2}$$

Check:

$$\frac{d}{dt}x(t) = \frac{1}{10t+1}(10) = 10(10t+1)^{-1} = v(t)$$

$$\frac{d}{dt}v(t) = 10(-1)(10t+1)^{-2}(10)$$

$$= -100(10t+1)^{-2} = a(t)$$

(2)(i) Show that $\int a(x)dx = \frac{1}{2}v^2$ (*)

(ii) Given that the acceleration of a particle as a function of its displacement is $a(x) = x + 1$, and that $x = 0$ and $v = 1$ when $t = 0$, find x in terms of t for $x > 0$ [The result (*) can be used.]

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Solution

$$(i) a = \frac{dv}{dt} = \frac{dv}{dx} \cdot \frac{dx}{dt} = v \frac{dv}{dx} \quad \text{and} \quad \frac{d}{dx} \left(\frac{1}{2} v^2 \right) = v \frac{dv}{dx}$$

Hence $a = \frac{d}{dx} \left(\frac{1}{2} v^2 \right)$, and so $\int a(x)dx = \frac{1}{2} v^2$, as required.

[Also, multiplying (*) by the mass m and applying limits gives

$$\int_{x_1}^{x_2} F(x)dx = \frac{1}{2} m v_2^2 - \frac{1}{2} m v_1^2, \text{ where } F(x) \text{ is force;}$$

ie the work-energy principle: work done = increase in kinetic energy]

$$(ii) \text{ From } (*), \frac{1}{2} v^2 = \int x + 1 dx = \frac{1}{2} x^2 + x + C$$

$$x = 0, v = 1 \Rightarrow C = \frac{1}{2}$$

$$\Rightarrow v^2 = x^2 + 2x + 1 = (x + 1)^2$$

$$\Rightarrow v = x + 1 \quad (v > 0 \text{ for } x > 0, \text{ as } a > 0)$$

$$\text{So } \frac{dx}{dt} = x + 1$$

$$\text{and hence } \int \frac{1}{x+1} dx = \int dt$$

$$\Rightarrow \ln(x + 1) = t + C \Rightarrow x + 1 = e^{t+C} \text{ or } x = Ae^t - 1$$

$$t = 0, x = 0 \Rightarrow A = 1, \text{ so that } x = e^t - 1$$

$$[\text{Check: } v = \frac{dx}{dt} = e^t = x + 1 \text{ and } a = \frac{dv}{dt} = e^t = x + 1]$$