Invariant Points & Lines - Introduction (4 pages; 16/4/20)

See also:

"Invariant Points & Lines - Conditions"

"Eigenvectors & Invariance"

"Invariant Points & Lines - Exercises"

(1) Lines of invariant points

(1.1) An invariant point of a transformation satisfies

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

This may only have the solution $x = y = 0$; ie when the Origin is the only invariant point.

For certain transformations however, there may be a line of invariant points.

It can be shown that lines of invariant points always pass through the Origin (see "Invariant Points & Lines - Conditions").

(1.2) The line of invariant points for a reflection in the line

$y = -x$ is the line itself. This can be verified, as follows:

$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$\Rightarrow -y = x$ and $-x = y$

These equations are consistent, and give $y = -x$ as the line of invariant points.

(2) Invariant lines

(2.1) An invariant line of a transformation (not to be confused with a line of invariant points) is a line such that any point on the line transforms to a point on the line (not necessarily a different point). A line of invariant points is thus a special case of an invariant line.

It can be shown that invariant lines don't necessarily pass through the Origin (see "Invariant Points & Lines - Conditions").

(2.2) The invariant lines for a reflection in the line $y = -x$ are:

(a) $y = -x$ (the line of invariant points), and

(b) all lines parallel to $y = x$

This can be verified, as follows:

Suppose that an invariant line has the equation $y = mx + k$.

[Strictly speaking, we should also consider lines of the form $x = \lambda$. This possibility is considered for the next example.]

Then the image of a point on this line is determined as follows:

$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ mx + k \end{pmatrix} = \begin{pmatrix} -mx - k \\ -x \end{pmatrix}$$

For this image to lie on the line, we require that

$m(-mx - k) + k = -x$

Also, this must hold for all points on the line; ie every x.

Hence, equating coefficients of x: -$m^2 = -1 \Rightarrow m = \pm 1$

and equating the constant terms: $-mk + k = 0$; ie $k(1 - m) = 0$

Then, if $m = 1$, the 2nd condition is satisfied for all values of k.

This gives the lines $y = x + k$ (ie (b) above).

If $m = -1$, the 2nd condition $\Rightarrow k = 0$

This gives the line $y = -x$ ((a) above).

(2.3) Example

(i) Find the line of invariant points for the shear represented by the matrix $\begin{pmatrix} 4 & -3 \\ 3 & -2 \end{pmatrix}$

[A necessary and sufficient condition for a shear $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ can be shown to be $ad - bc = 1$ and $a + d = 2$.]

(ii) Find the other invariant lines of the shear, using a matrix method.

Solution

(i) $\begin{pmatrix} 4 & -3 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow 4x - 3y = x$ and $3x - 2y = y$

ie $y = x$

(ii) Suppose that an invariant line has the equation $y = mx + k$

[The possibility of $x = \lambda$ is considered at the end.]

The image of a point on this line is:

$$\begin{pmatrix} 4 & -3 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} x \\ mx + k \end{pmatrix} = \begin{pmatrix} 4x - 3mx - 3k \\ 3x - 2mx - 2k \end{pmatrix}$$

For this image to lie on the line, we require that

$m(4x - 3mx - 3k) + k = 3x - 2mx - 2k$

Then, equating coefficients of x: $4m - 3m^2 = 3 - 2m$

$\Rightarrow 3m^2 - 6m + 3 = 0$ or $m^2 - 2m + 1 = 0$, so that $(m-1)^2 = 0$ and hence $m = 1$

Equating the constant terms: $-3mk + k = -2k$

$\Rightarrow 3k(1-m) = 0$, so that either $k = 0$ *or* $m = 1$

Combining the two conditions gives: $m = 1$ and k can take any value; ie the invariant lines are of the form $y = x + k$.

This is to be expected for a shear, where the invariant lines are all the lines parallel to the shear line (which is the line of invariant points).

[Considering lines of the form $x = \lambda$:

$$\begin{pmatrix} 4 & -3 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} \lambda \\ y \end{pmatrix} = \begin{pmatrix} 4\lambda - 3y \\ 3\lambda - 2y \end{pmatrix}$$

For this image to lie on the line, we require $4\lambda - 3y = \lambda$,

and this is only true for $y = \lambda$; ie the single point $\begin{pmatrix} \lambda \\ \lambda \end{pmatrix}$. So there is no invariant **line** of the form $x = \lambda$.]