Invariant Points & Lines - Exercises (Sol'ns)

(3 pages; 16/4/20)

(1) Find the value of k for which the transformation $\begin{pmatrix} 2 & 4 \\ 3 & k \end{pmatrix}$ has a line of invariant points, and find this line.

Solution

Suppose that $\begin{pmatrix} 2 & 4 \\ 3 & k \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix}$

Then $2p + 4q = p$ and $3p + kq = q$

so that $4q = -p$ & $(k-1)q = -3p$

Hence $\frac{q}{p} = -\frac{1}{4}$ & $\frac{q}{p} = -\frac{3}{k-1}$

so that $-\frac{1}{4} = -\frac{3}{k-1} \Rightarrow k - 1 = 12$ & hence $k = 13$

[See "Invariant Points & Lines - Conditions". A line of invariant points will exist when $trM = |M| + 1$; in this case, when $2 + k = 2k - 12 + 1$]

To find the line:

$\begin{pmatrix} 2 & 4 \\ 3 & 13 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix} \Rightarrow 2p + 4q = p$ & $3p + 13q = q$

so that $4q = -p$ (or $12q = -3p$)

and hence $q = -\frac{p}{4}$

ie the invariant points lie on the line $y = -\frac{x}{4}$

Check: $\begin{pmatrix} 2 & 4 \\ 3 & 13 \end{pmatrix}\begin{pmatrix} 4 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$

(2) (i) Use a matrix method to find the invariant lines for a reflection in the y-axis.

(ii) Investigate the invariant lines for a reflection in the x-axis.

Solution

(i) Suppose that an invariant line has the equation $y = mx + c$ (noting that lines of the form $x = a$ aren't invariant lines)

The image of a point on this line is:

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ mx + c \end{pmatrix} = \begin{pmatrix} -x \\ mx + c \end{pmatrix}$$

For this image to lie on the line, we require that

$m(-x) + c = mx + c$

$\Rightarrow 2mx = 0$

$\Rightarrow m = 0$ (for any value of c), or $x = 0$

ie the invariant lines are $y = c$ and $x = 0$ (the line of invariant points)

(ii) Suppose that an invariant line has the equation $y = mx + c$.

The image of a point on this line is:

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ mx + c \end{pmatrix} = \begin{pmatrix} x \\ -mx - c \end{pmatrix}$$

For this image to lie on the line, we require that

$mx + c = -mx - c$

Equating coefficients of x: $m = -m \Rightarrow m = 0$

Equating the constant terms: $c = -c \Rightarrow c = 0$

So we have only found the line $y = 0$ (the line of invariant points).

Now consider lines of the form $x = a$.

This gives $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a \\ y \end{pmatrix} = \begin{pmatrix} a \\ -y \end{pmatrix}$

As this lies on the line $x = a$ for all values of a, the lines $x = a$ are also invariant lines.