Invariant Points & Lines - Conditions (12 pages; 16/4/20)

See also:

"Invariant Points & Lines - Introduction"

"Eigenvectors & Invariance"

Contents

(A) Lines of the form $y = mx + k$

(A.1) Invariant lines

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} p \\ mp + k \end{pmatrix} = \begin{pmatrix} ap + c(mp + k) \\ bp + d(mp + k) \end{pmatrix}$$

and so $bp + d(mp + k) = m\{ap + c(mp + k)\} + k$ for all p

Equating coefficients of p: $b + dm = am + cm^2$

$\Rightarrow cm^2 + (a - d)m - b = 0$ (1)

Equating coefficients of p^0: $dk = mck + k$

$\Rightarrow k(d - mc - 1) = 0$ (2)

Case 1: $c = 0, a \neq d$

(1) $\Rightarrow m = \frac{b}{a-d}$

(2) $\Rightarrow k = 0$ or $d = 1$

So, when $d = 1$, there are invariant lines $y = \frac{b}{a-1}x + k$

When $d \neq 1$, there is a single invariant line $y = \frac{b}{a-d}x$

Case 2: $\boldsymbol{c = 0, a = d}$

No solution unless $b = 0$

When $b = 0$, (2) $\Rightarrow k(d-1) = 0$

When $d = 1$, M is the identity matrix - ie a trivial case.

When $d \neq 1$, there are invariant lines $y = mx$ (for any m)

(M represents an enlargement)

Case 3: $\boldsymbol{c \neq 0}$

For there to be a solution to (1), $(a-d)^2 - 4c(-b) \geq 0$;

ie $(a+d)^2 - 4ad + 4bc \geq 0$

ie $(trM)^2 \geq 4|M|$, where the trace of M, trM is defined as $a + d$

[For the general matrix $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$,

$(trM)^2 - 4|M| = (a+d)^2 - 4(ad - bc) = (a-d)^2 + 4bc$,

so this condition is satisfied whenever b & c have the same sign.]

When this condition is satisfied, there will be two invariant lines through the Origin: $y = m_1 x$ & $y = m_2 x$

There will be one line when $(a-d)^2 - 4c(-b) = 0$;

ie $(a+d)^2 - 4ad + 4bc = 0$, so that $(trM)^2 = 4|M|$

For there to be an invariant line that doesn't pass through the Origin, (2) $\Rightarrow m = \frac{d-1}{c}$

Then, from (1), $m = \frac{d-a\pm\sqrt{(a-d)^2+4cb}}{2c}$,

so that $\frac{d-1}{c} = \frac{d-a\pm\sqrt{(a-d)^2+4cb}}{2c}$

$\Rightarrow 2(d-1) - d + a = \pm\sqrt{(a-d)^2 + 4cb}$

$\Rightarrow (d + a - 2)^2 = (a-d)^2 + 4bc$

$\Rightarrow d^2 + a^2 + 4 + 2ad - 4d - 4a = a^2 + d^2 - 2ad + 4bc$

$\Rightarrow 4 - 4d - 4a = -4ad + 4bc$

$\Rightarrow 1 - tr(M) = -|M|$

ie $tr(M) = |M| + 1$

(A.2) Lines of invariant points must pass through the Origin

(considering lines of the form $y = mx + k$ for the moment)

Proof

Suppose that $\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} x \\ mx+k \end{pmatrix} = \begin{pmatrix} x \\ mx+k \end{pmatrix}$, for all x,

where $k \neq 0$ (so that the line of invariant points is $y = mx + k$).

Then $ax + c(mx + k) = x$ & $bx + d(mx + k) = mx + k$

Equating coefficients of x: $a + cm = 1$ & $b + dm = m$

Equating coefficients of x^0: $ck = 0$ & $dk = k$

As $k \neq 0$, this leads to $c = 0, d = 1, a = 1$ & $b = 0$;

ie $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ is the identity matrix.

(A.3) Lines of invariant points

Suppose that there is a line of invariant points $y = mx$,

so that $\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} x \\ mx \end{pmatrix} = \begin{pmatrix} x \\ mx \end{pmatrix}$ for all x

ie $\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} x \\ mx \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ mx \end{pmatrix}$

or $\begin{pmatrix} a-1 & c \\ b & d-1 \end{pmatrix}\begin{pmatrix} x \\ mx \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

For there to be a solution other than $x = 0, y = 0$,

$\begin{vmatrix} a-1 & c \\ b & d-1 \end{vmatrix} = 0$

$\Rightarrow (a-1)(d-1) - bc = 0$

$\Rightarrow 1 - (a+d) + ad - bc = 0$

$\Rightarrow trM = |M| + 1$

[As lines of invariant points are special cases of invariant lines, we expect that $(trM)^2 \geq 4|M|$:

$trM = |M| + 1 \Rightarrow (trM)^2 - 4|M| = (|M| + 1)^2 - 4|M|$

$= (1 - |M|)^2 \geq 0$]

(A.4) Eigenvalue approach (for lines passing through the Origin)

Suppose that $\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} x \\ mx \end{pmatrix} = \lambda\begin{pmatrix} x \\ mx \end{pmatrix}$ for all x

[If $\lambda = 1$, then $y = mx$ will be a line of invariant points; otherwise it will be an invariant line.]

Then $\begin{pmatrix} a-\lambda & c \\ b & d-\lambda \end{pmatrix}\begin{pmatrix} x \\ mx \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

For this to have a solution other than $x = 0$,

$\begin{vmatrix} a-\lambda & c \\ b & d-\lambda \end{vmatrix} = 0,$

so that $(a-\lambda)(d-\lambda) - bc = 0$

ie $\lambda^2 - (a+d)\lambda + ad - bc = 0.$

For λ to exist, $(a+d)^2 - 4(ad-bc) \geq 0$;

ie $(trM)^2 \geq 4|M|$, as before

(B) Lines of the form $x = \lambda$

(B.1) Invariant lines

Suppose that $\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} \lambda \\ y \end{pmatrix} = \begin{pmatrix} \lambda \\ y' \end{pmatrix}$, for all y.

Then $a\lambda + cy = \lambda$ for all y,

so that $c = 0$

Then $x = 0$ is always an invariant line.

If $a = 1$, then $x = \lambda$ is an invariant line, for all λ.

(B.2) Lines of invariant points

Suppose that $\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} \lambda \\ y \end{pmatrix} = \begin{pmatrix} \lambda \\ y \end{pmatrix}$, for all y.

Then $a\lambda + cy = \lambda \Rightarrow c = 0$ & either $\lambda = 0$ or $a = 1$

and $b\lambda + dy = y \Rightarrow d = 1$ & either $b = 0$ or $\lambda = 0$

$\boldsymbol{\lambda \neq 0}$

$\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ is the identity matrix

So, for lines of the form $x = \lambda$ as well, lines of invariant points have to pass through the origin (excluding the trivial case where all lines are lines of invariant points).

$\boldsymbol{\lambda = 0}$

$\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix}$

(C) Conclusions

(C.1) Lines of invariant points must pass through the Origin.

(C.2) For transformations $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ where $c \neq 0$, there will be a single invariant line (of the form $y = mx + k$) when

$(trM)^2 = 4|M|$, and two such invariant lines when

$(trM)^2 > 4|M|$ (and this condition is satisfied whenever b & c have the same sign).

(C.3) When $trM = |M| + 1$, there will be invariant lines that don't pass through the Origin, and there will also be a line of invariant points of the form $y = mx$. The line of invariant points belongs to the family of invariant lines: they have the same gradient.

A shear is an example of this ($|M| = 1$ & $trM = 2$).

(D) Examples of cases

Note: All lines of invariant points are invariant lines.

(1) $c = 0, a \neq d, d \neq 1$; eg $\begin{pmatrix} 2 & 0 \\ 3 & 4 \end{pmatrix}$

$y = \frac{b}{a-d}x$ & $x = 0$ are invariant lines

Check

$\begin{pmatrix} 2 & 0 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} x \\ -\frac{3}{2}x \end{pmatrix} = \begin{pmatrix} 2x \\ 3x - 6x \end{pmatrix} = \begin{pmatrix} 2x \\ -3x \end{pmatrix} = 2\begin{pmatrix} x \\ -\frac{3}{2}x \end{pmatrix}$,

so that $y = \frac{3}{2-4}x$ is an invariant line [$\begin{pmatrix} 1 \\ -\frac{3}{2} \end{pmatrix}$, or any multiple of it, is an eigenvector, with eigenvalue 2].

And a point of the form $\begin{pmatrix} 0 \\ p \end{pmatrix} = p\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ will be mapped to $p\begin{pmatrix} 0 \\ 4 \end{pmatrix}$, so that $x = 0$ is an invariant line also.

(2) $c = 0, a \neq d, d = 1$; eg $\begin{pmatrix} 2 & 0 \\ 3 & 1 \end{pmatrix}$

$y = \frac{b}{a-1}x + k$ are invariant lines

and $x = 0$ is a line of invariant points

Check

$\begin{pmatrix} 2 & 0 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} x \\ 3x + k \end{pmatrix} = \begin{pmatrix} 2x \\ 3x + 3x + k \end{pmatrix} = \begin{pmatrix} 2x \\ 3(2x) + k \end{pmatrix}$,

so that $y = 3x + k$ are invariant lines.

And a point of the form $\begin{pmatrix} 0 \\ p \end{pmatrix} = p\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ will be mapped to

$p\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ p \end{pmatrix}$, so that $x = 0$ is a line of invariant points.

(3) $c = 0, a = d \neq 1, b = 0$; eg $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

$y = mx$ (for all m) & $x = 0$ are invariant lines

Check

$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ mx \end{pmatrix} = \begin{pmatrix} 2x \\ 2mx \end{pmatrix} = \begin{pmatrix} 2x \\ m(2x) \end{pmatrix}$,

so that $y = mx$ (for all m) are invariant lines

(4) $c = 0, a = 1, d \neq 1$; eg $\begin{pmatrix} 1 & 0 \\ 3 & 4 \end{pmatrix}$

$y = \frac{b}{a-d}x$ & $x = \lambda$ (for all λ) are invariant lines

Check

$\begin{pmatrix} 1 & 0 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} \lambda \\ y \end{pmatrix} = \begin{pmatrix} \lambda \\ y' \end{pmatrix}$,

so that $x = \lambda$ (for all λ) are invariant lines

(5) $c = 0, a = 1, d = 1$; eg $\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$

shear in the y-direction

$x = \lambda$ (for all λ) are invariant lines

$x = 0$ is a line of invariant points

Check

Consider $\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} x \\ mx + k \end{pmatrix} = \begin{pmatrix} x \\ 3x + mx + k \end{pmatrix}$

If $y = mx + k$ is an invariant line,

then $3x + mx + k = mx + k$,

but this is impossible.

(6) $c = 0, a \neq 1, d = 1, b = 0$; eg $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$

stretch in the x-direction

$y = \frac{b}{a-1}x + k = k$ are invariant lines

$x = 0$ is a line of invariant points

(7) $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

reflection in the y-axis

$y = \frac{b}{a-1}x + k = k$ are invariant lines;

$x = 0$ is a line of invariant points

(8) $c \neq 0,\ (trM)^2 > 4|M|$ (eg $bc > 0$); eg $\begin{pmatrix} 2 & -4 \\ -3 & 5 \end{pmatrix}$

$y = m_1 x$ & $y = m_2 x$ are invariant lines

Check

$\begin{pmatrix} 2 & -4 \\ -3 & 5 \end{pmatrix}\begin{pmatrix} x \\ mx + k \end{pmatrix} = \begin{pmatrix} 2x - 4mx - 4k \\ -3x + 5mx + 5k \end{pmatrix}$

Then $-3x + 5mx + 5k = m(2x - 4mx - 4k) + k$ for all x

$\Rightarrow -3 + 5m = 2m - 4m^2$ (equating coeffs of x)

$\Rightarrow 4m^2 + 3m - 3 = 0$

$\Rightarrow m = \frac{-3 \pm \sqrt{57}}{8}$

& $5k = -4mk + k$ (equating coeffs of x^0)

$\Rightarrow k = 0$ or $m = -1$

So $m = \frac{-3\pm\sqrt{57}}{8}$ and $k = 0$

and the invariant lines are $y = m_1 x$ & $y = m_2 x$

(9) $c \neq 0,\ trM = |M| + 1$; eg $\begin{pmatrix} 2 & -4 \\ -3 & 13 \end{pmatrix}$

$y = m_1 x + k$ are invariant lines

$y = m_2 x$ is a line of invariant points

Check

$$\begin{pmatrix} 2 & -4 \\ -3 & 13 \end{pmatrix}\begin{pmatrix} x \\ mx + k \end{pmatrix} = \begin{pmatrix} 2x - 4mx - 4k \\ -3x + 13mx + 13k \end{pmatrix}$$

Then $-3x + 13mx + 13k = m(2x - 4mx - 4k) + k$ for all x

$\Rightarrow -3 + 13m = 2m - 4m^2$ (equating coeffs of x)

$\Rightarrow 4m^2 + 11m - 3 = 0$

$\Rightarrow (4m - 1)(m + 3) = 0$

$\Rightarrow m = \frac{1}{4}$ or -3

& $13k = -4mk + k$ (equating coeffs of x^0)

$\Rightarrow k = 0$ or $m = -3$

So the invariant lines are $y = \frac{1}{4}x,\ y = -3x + k$

For invariant points:

$$\begin{pmatrix} 2 & -4 \\ -3 & 13 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$\Rightarrow 2x - 4y = x$ (or $-3x + 13y = y$)

$\Rightarrow y = \frac{1}{4}x$

(10) $c \neq 0,\ (trM)^2 = 4|M|$; eg $\begin{pmatrix} 2 & 2 \\ -2 & 6 \end{pmatrix}$

Single invariant line: $y = mx$ (for some m).

Check

For an invariant line of the form $y = mx + k$:

$$\begin{pmatrix} 2 & 2 \\ -2 & 6 \end{pmatrix}\begin{pmatrix} x \\ mx + k \end{pmatrix} = \begin{pmatrix} 2x + 2mx + 2k \\ -2x + 6mx + 6k \end{pmatrix}$$

We require $-2x + 6mx + 6k = m(2x + 2mx + 2k) + k$

Equating coeffs of x: $-2 + 6m = 2m + 2m^2$

$\Rightarrow 2m^2 - 4m + 2 = 0$

$\Rightarrow m^2 - 2m + 1 = 0$

$\Rightarrow (m - 1)^2 = 0 \Rightarrow m = 1$

Equating coeffs of x^0: $6k = 2mk + k$

$\Rightarrow k = 0$ or $m = \frac{5}{2}$

Hence $m = 1$ & $k = 0$, and the single invariant line is $y = x$

For invariant points:

$$\begin{pmatrix} 2 & 2 \\ -2 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$\Rightarrow 2x + 2y = x \Rightarrow y = -\frac{x}{2}$

and $-2x + 6y = y \Rightarrow y = \frac{2x}{5}$

So there is no solution (note that $trM \neq |M| + 1$).