Integration Exercises - Part 3 (Sol'ns) (Hyperbolic Functions)

(12 pages; 17/4/20)

(The constant of integration has been omitted throughout.)

(1) $\int cosech^2x \ dx$

Solution

Given that $\frac{d}{dx} \ tanhx = \ sech^2x$, we could investigate $\frac{d}{dx} \ cothx = \frac{d}{dx}\frac{coshx}{sinhx} = \frac{sinhx.sinhx - coshx.coshx}{sinh^2x} = \frac{-1}{sinh^2x} = -cosech^2x$

so that $\int cosech^2x \ dx = -cothx + c$

(2) $\int tanhx \ dx$ (from 1st principles)

Solution

$\int tanhx \ dx = \int \frac{sinhx}{coshx} dx$

As $\int sinhx \ dx = coshx$, let $u = coshx$

Then $du = sinhx \ dx$, and $\int \frac{sinhx}{coshx} dx = \int \frac{1}{u} du = \ln|u| + c$

$= \ln(coshx) + \mathrm{c}$

[No moduli signs are needed, as $coshx > 0$ for all x]

(3) $\int sechxtanhx \ dx$

Solution

$\int sechxtanhx \ dx = \int \frac{sinhx}{cosh^2x} dx$

As $\int sinhx \ dx = coshx$, let $u = coshx$

Then $du = sinhx \ dx$, and $\int \frac{sinhx}{cosh^2x} dx = \int \frac{1}{u^2} du = -\frac{1}{u} + c$

$= -sechx + c$

(4) $\int tanh^2x\ dx$

Solution

$\int tanh^2x\ dx = \int 1 - sech^2x\ dx$

$= x - tanhx + c$ (from differentiation table)

(5) $\int sech^2x\ tanh^2x\ dx$

Solution

As $\int sech^2x\ dx = tanhx$, let $u = tanhx$,

so that $du = sech^2x\ dx$,

and $\int sech^2x\ tanh^2x\ dx = \int u^2du = \frac{1}{3}u^3 + c = \frac{1}{3}tanh^3x + c$

(6) $\int sech^3x\ tanhx\ dx$

Solution

As we have a power of $sechx$, consider $\frac{d}{dx}sechx = -sechxtanhx$,

from (i).

Then let $u = sechx$, so that $du = -sechxtanhx\ dx$

and $\int sech^3x\ tanhx\ dx = -\int u^2du =$

$= -\frac{1}{3}u^3 + c = -\frac{1}{3}sech^3x + c$

(7) $\int cosh^2x\, dx$

Solution

$cosh^2x - sinh^2x = 1$ & $cosh^2x + sinh^2x = cosh2x$,

so that $cosh^2x = \frac{1}{2}(1 + cosh2x)$,

and $\int cosh^2x\, dx = \frac{1}{2}\int 1 + cosh2x\, dx = \frac{x}{2} + \frac{1}{4}sinh2x + c$

(8) $\int cosh^3x\, dx$

Solution

$\int cosh^3x\, dx = \int coshx(1 - sinh^2x)\, dx$

Let $u = sinhx$, so that $du = coshx\, dx$,

and $\int coshx(1 - sinh^2x)\, dx = \int 1 - u^2\, du$

$= u - \frac{1}{3}u^3 + c = sinhx - \frac{1}{3}sinh^3x + c$

(9) $\int sechx\, dx$

Solution

Method 1

$\int sechx\, dx = \int \frac{coshx}{cosh^2x} dx = \int \frac{coshx}{1+sinh^2x}\, dx$

Let $u = sinhx$, so that $du = coshx\, dx$

Then $\int \frac{coshx}{1+sinh^2x}\, dx = \int \frac{1}{1+u^2}\, du$

$= \arctan(sinhx) + c$

Method 2

$\int sechxdx = \int \frac{2}{(e^x+e^{-x})} dx = 2\int \frac{e^x}{e^{2x}+1} dx$

Let $u = e^x$, so that $du = e^x dx$

and $2\int\frac{e^x}{e^{2x}+1}dx \ = 2\int\frac{1}{u^2+1}du$

$= 2\,\arctan(e^x) + c'$

To show that $\arctan(sinhx) + c = 2\arctan(e^x) + c'$:

Let $\theta = \arctan(sinhx)$ and $\phi = 2\arctan(e^x)$

We need to show that θ & ϕ differ by a constant.

$tan\theta = sinhx$ & $tan\left(\frac{\phi}{2}\right) = e^x$

Let $t = tan\left(\frac{\phi}{2}\right)$

Then $sinhx = \frac{1}{2}(e^x - e^{-x}) = \frac{1}{2}\left(t - \frac{1}{t}\right) = \frac{t^2-1}{2t}$

[We can now use the standard right-angled triangle with sides $1 - t^2, 2t$ & $1 + t^2$: $tan\phi = \frac{2t}{1-t^2}$ & the hypotenuse follows from Pythagoras.]

So $sinhx = -\tan\left(\frac{\pi}{2} - \frac{\phi}{2}\right) = \tan(\frac{\phi}{2} - \frac{\pi}{2})$

Thus $\tan\left(\frac{\phi}{2} - \frac{\pi}{2}\right) = tan\theta$, and so $\frac{\phi}{2} - \frac{\pi}{2}$ & ϕ differ by a multiple of π; ie θ & ϕ differ by a constant, as required.

(10) $\int\frac{1}{\sqrt{4x^2-9}}dx$

Solution

$\int\frac{1}{\sqrt{4x^2-9}}dx = \frac{1}{2}\int\frac{1}{\sqrt{x^2-\left(\frac{3}{2}\right)^2}}\,dx$

$= \frac{1}{2}\,arcosh\left(\frac{x}{\left(\frac{3}{2}\right)}\right) + C = \frac{1}{2}\,arcosh\left(\frac{2x}{3}\right) + C$

(11) (a)$\int cosh^{-1}x\ dx$ (b)$\int sinh^{-1}x\ dx$ (c)$\int tanh^{-1}x\ dx$

Solution

(a) By Parts, $\int cosh^{-1}x\ dx = xcosh^{-1}x - \int x\frac{1}{\sqrt{x^2-1}}\ dx$

$= xcosh^{-1}x - \frac{1}{2}\int\frac{2x}{\sqrt{x^2-1}}\ dx$

$= xcosh^{-1}x - (\frac{1}{2})\frac{\sqrt{x^2-1}}{(\frac{1}{2})}$

$= xcosh^{-1}x - \sqrt{x^2-1}$

Alternative method

Let $u = cosh^{-1}x$, so that $coshu = x$ and $sinhu.du = dx$

Then $\int cosh^{-1}x\ dx = \int u\ sinhu\ du$

$= ucoshu - \int coshu\ du$

$= ucoshu - sinhu$

$= cosh^{-1}x.x - \sqrt{cosh^2u - 1}$

$= cosh^{-1}x.x - \sqrt{x^2-1}$

[Note: As $u = cosh^{-1}x > 0, sinhu > 0$ and so

$sinhu = +\sqrt{cosh^2u - 1}$]

(b) By Parts, $\int sinh^{-1}x\ dx = xsinh^{-1}x - \int x\frac{1}{\sqrt{x^2+1}}\ dx$

$= xsinh^{-1}x - \frac{1}{2}\int\frac{2x}{\sqrt{x^2+1}}\ dx$

$= xsinh^{-1}x - (\frac{1}{2})\frac{\sqrt{x^2+1}}{(\frac{1}{2})}$

$= xsinh^{-1}x - \sqrt{x^2+1}$

(c) By Parts, $\int tanh^{-1}x\ dx = xtanh^{-1}x - \int x\frac{1}{1-x^2}\ dx$

$= xtanh^{-1}x + \frac{1}{2}\int\frac{-2x}{1-x^2}\ dx$

$= xtanh^{-1}x + \frac{1}{2}\ln(1-x^2)$

(12) $\int\frac{1}{(x^2-9)^{\frac{3}{2}}}\ dx$

Solution

Let $x = 3coshu$, so that $dx = 3sinhu\ du$

Then $I = \int\frac{3sinhu}{27sinh^3u}\ du = \frac{1}{9}\int cosech^2u\ du$

As $\frac{d}{du}tanhu = sech^2u$, we can investigate $\frac{d}{du}cothu$:

$$\frac{d}{du}\frac{coshu}{sinhu} = \frac{sinhusinhu - coshucoshu}{sinh^2u} = -\frac{1}{sinh^2u} = -cosech^2u$$

Thus $I = -\frac{1}{9}cothu = -\frac{1}{9}\frac{coshu}{\sqrt{cosh^2u-1}} = -\frac{1}{9}\frac{\frac{x}{3}}{\sqrt{\frac{x^2}{9}-1}} = -\frac{1}{9}\frac{x}{\sqrt{x^2-9}}$

Alternative method

Let $x = 3secu$, so that $dx = 3secutanu$

Then $I = \int\frac{3secutanu}{27tan^3u}\ du = \frac{1}{9}\int\frac{1/cosu}{\left(\frac{sinu}{cosu}\right)^2}\ du = \frac{1}{9}\int\frac{cosu}{sin^2u}\ du$

$= \frac{1}{9}\frac{1}{sinu}(-1) = -\frac{1}{9}\frac{1}{\sqrt{1-cos^2u}} = -\frac{1}{9}\frac{1}{\sqrt{1-(3/x)^2}} = -\frac{1}{9}\frac{x}{\sqrt{x^2-9}}$

(13) $\int\sqrt{4+x^2}\ dx$

Solution

Let $x = 2sinhu$, so that $dx = 2coshu\ du$

and $I = 2\int \sqrt{1 + sinh^2u}\,(2coshu)du = 4\int cosh^2u\; du$

$= 2\int 1 + \cosh(2u)\; du = 2\left(u + \frac{1}{2}\sinh(2u)\right)$

$= 2sinh^{-1}\left(\frac{x}{2}\right) + 2sinhucoshu$

$= 2sinh^{-1}\left(\frac{x}{2}\right) + \; x\sqrt{\left(\frac{x}{2}\right)^2 + 1}$

$= 2sinh^{-1}\left(\frac{x}{2}\right) + \; x\sqrt{\left(\frac{x}{2}\right)^2 + 1}$

$= 2sinh^{-1}\left(\frac{x}{2}\right) + \frac{1}{2}x\sqrt{x^2 + 4}$

[Note: The alternative substitution $x = 2tanu$, leads to a much more complicated expression.]

(14) $\int \sqrt{\frac{x}{1+x}}\;\; dx$

Solution

Let $x = sinh^2u$, so that $dx = 2sinhucoshu\; du$

and $I = \int \frac{sinhu}{coshu}\,(\,2sinhucoshu)\; du$

$= 2\int sinh^2u\;\; du = \int (\cosh(2u) - 1)\; du$

$= \frac{1}{2}\sinh(2u) - u$

$= sinhu\; coshu - u$

$= \sqrt{x(1 + x)} - sinh^{-1}(\sqrt{x})$

(15) $\int \sqrt{4x^2 - 1}\; dx$

Solution

Let $2x = coshu$, so that $2\; dx = sinhu\; du$

and $I = \frac{1}{2}\int sinhu\,.sinhu\ du = \frac{1}{4}\int \cosh(2u) - 1\ du$

$= \frac{1}{8}\sinh(2u) - \frac{1}{4}u = \frac{1}{4}sinhucoshu - \frac{1}{4}cosh^{-1}(2x)$

$= \frac{1}{2}x\sqrt{4x^2 - 1} - \frac{1}{4}cosh^{-1}(2x)$

(16) $\int x^2\sqrt{1 + x^2}\ dx$

Solution

Let $x = sinhu$, so that $dx = coshu\ du$

and $I = \int sinh^2u\ coshu\ coshu\ du$

$= \int sinh^2u(sinh^2u + 1)du$

[A reduction formula could be found for $\int sinh^4u\ \ du$ at this point, as an alternative to the following.]

$= \int \frac{1}{2}[\cosh(2u) - 1]\left\{\frac{1}{2}[\cosh(2u) - 1] + 1\right\}du$

$= \frac{1}{4}\int[\cosh(2u) - 1][\cosh(2u) + 1]du$

$= \frac{1}{4}\int cosh^2(2u) - 1\ du$

$= \frac{1}{4}\int \frac{1}{2}[(\cosh(4u) + 1] - 1\ du$

$= \frac{1}{32}\sinh(4u) - \frac{u}{8}$

$= \frac{1}{16}\sinh(2u)\cosh(2u) - \frac{u}{8}$

$= \frac{1}{8}sinhucoshu(cosh^2u + sinh^2u) - \frac{u}{8}$

$= \frac{1}{8}x\sqrt{x^2 + 1}(1 + 2x^2) - \frac{1}{8}sinh^{-1}x$

(17) $\int \frac{1}{\sqrt{x(x+1)}} dx$

Solution

$x(x+1) = x^2 + x = (x+\frac{1}{2})^2 - \frac{1}{4}$

So $I = \int \frac{1}{\sqrt{(x+\frac{1}{2})^2 - \frac{1}{4}}} dx$

$= cosh^{-1}\left(\frac{x+\frac{1}{2}}{\frac{1}{2}}\right) = cosh^{-1}(2x+1)$

(18) $\int \frac{x^2}{\sqrt{x^6-1}} dx$

Solution

Noting that $\int x^2 dx = \frac{1}{3}x^3$, and that $x^6 = (x^3)^2$, let $u = x^3$, so that $du = 3x^2\, dx$

and $I = \frac{1}{3}\int \frac{1}{\sqrt{u^2-1}}\, du = \frac{1}{3} cosh^{-1}(x^3)$

(19) $\int_0^4 \frac{4x+1}{\sqrt{x^2+9}} dx$

Solution

$I = 2\int_0^4 \frac{2x}{\sqrt{x^2+9}} dx + \left[arsinh\left(\frac{x}{3}\right)\right]_0^4$

$= 2\left[\frac{\sqrt{x^2+9}}{\left(\frac{1}{2}\right)}\right]_0^4 + arsinh(\frac{4}{3})$

$= 4(5-3) + \ln(\frac{4}{3} + \sqrt{\left(\frac{4}{3}\right)^2 + 1})$

$= 8 + \ln(\frac{1}{3}(4+5))$

$= 8 + ln3$

(20) $\int_0^1 \sqrt{16x^2+9}\,dx$

Solution

Let $4x = 3sinhu$, so that $4dx = 3coshu\,du$

$I = 3\int_0^1 coshu\,.\frac{3}{4}coshu\,du$

$= \frac{9}{4}\int_0^1 \frac{1}{2}(1+cosh2u)du$

[since $cosh^2u - sinh^2u = 1$ & $cosh^2u + sinh^2u = cosh2u$]

$= \frac{9}{8}[u + \frac{1}{2}sinh2u]_0^1$

$= \frac{9}{8}\left(1 + \frac{1}{4}(e^2 - e^{-2})\right) = \frac{9}{8} + \frac{9}{32}(e^2 - e^{-2})$

(21) $\int \sqrt{\frac{x+1}{x}}\ dx$

Solution

Let $x = sinh^2y$ [so that $x + 1 = cosh^2y$]

Then $dx = 2sinhycoshy\,dy$

and $I = \int \frac{coshy}{sinhy}\,.2sinhycoshy\,dy$

$= 2\int cosh^2y\ dy$

$= \int 1 + cosh2y\,dy$

$= y + \frac{1}{2}sinh2y$

$= arsinh(\sqrt{x}) + sinhycoshy$

$= arsinh(\sqrt{x}) + \sqrt{x(x+1)}$

(22) $\int \frac{1}{(2x^2+3)^{\frac{3}{2}}}\ dx$

Solution

Method 1

Let $2x^2 = 3sinh^2y$, so that $4x\ dx = 6sinhycoshy\ dy$

Then $I = \int \frac{1}{3^{\frac{3}{2}}(sinh^2y+1)^{\frac{3}{2}}} \frac{6sinhycoshy\ dy}{4x}$

$= \frac{3}{2}\int \frac{sinhycoshy}{3^{\frac{3}{2}}cosh^3ysinhy\sqrt{\frac{3}{2}}} dy$

$= \sqrt{\frac{3}{2}}\int \frac{sech^2y}{3^{\frac{3}{2}}}\ dy$

$= 3^{-1}2^{-\frac{1}{2}}tanhy$

$= \frac{1}{3\sqrt{2}}tanhy$

As $2x^2 = 3sinh^2y,\ cosh^2y = \frac{2}{3}x^2 + 1,$

so that $sech^2y = \frac{1}{\frac{2}{3}x^2+1} = \frac{3}{2x^2+3}$

and $-tanh^2y = sech^2y - 1$ [applying Osborn's rule]

so that $\frac{1}{3\sqrt{2}}tanhy = \frac{1}{3\sqrt{2}}\sqrt{1 - \frac{3}{2x^2+3}}$

$= \frac{1}{3\sqrt{2}}\sqrt{\frac{2x^2}{2x^2+3}} = \frac{x}{3\sqrt{2x^2+3}}$

Method 2

Let $2x^2 = 3tan^2\theta$, so that $4x\ dx = 6tan\theta sec^2\theta\ d\theta$

Then $I = \int \frac{1}{3^{\frac{3}{2}}(tan^2\theta+1)^{\frac{3}{2}}} \frac{6tan\theta sec^2\theta}{4x}\ d\theta$

$= \frac{3}{2}\int \frac{1}{3^{\frac{3}{2}}sec^3\theta} \frac{tan\theta sec^2\theta}{tan\theta\sqrt{\frac{3}{2}}}\ d\theta$

$= 3^{-1}2^{-\frac{1}{2}}\int cos\theta\ d\theta$

$= \frac{1}{3\sqrt{2}}sin\theta$

As $2x^2 = 3tan^2\theta, sec^2\theta = \frac{2}{3}x^2 + 1,$

so that $cos^2\theta = \frac{1}{\frac{2}{3}x^2+1} = \frac{3}{2x^2+3}$

and $\frac{1}{3\sqrt{2}}sin\theta = \frac{1}{3\sqrt{2}}\sqrt{1 - \frac{3}{2x^2+3}}$

$= \frac{1}{3\sqrt{2}}\sqrt{\frac{2x^2}{2x^2+3}} = \frac{x}{3\sqrt{2x^2+3}}$

(23) $\int cosech(3x)\coth(3x)\ dx$

Solution

Note that $\frac{d}{dx}(cosecx) = -cosecx\ cotx$

So suppose that the solution is of the form $Acosech(3x)$.

Then $\frac{d}{dx}\left(Acosech(3x)\right) = A\frac{d}{dx}\left(\left(sinh(3x)\right)^{-1}\right)$

$= A(-1)\left(sinh(3x)\right)^{-2}(3cosh(3x))$

$= -3Acoth(3x)cosech(3x)$

So we require $A = -\frac{1}{3}$

and $I = -\frac{1}{3}cosech(3x)$