Integration Exercises - Part 2 (Sol'ns)(22 pages; 8/8/19)

(Note: The constant of integration has been omitted throughout.)

(1) $\int \frac{1}{1+e^x} dx$

Solution

Step 1: rearrange

$I = \int \frac{e^{-x}}{e^{-x}+1} dx$

Step 2: substitute $u = e^{-x} + 1$, so that $du = -e^{-x}dx$

$I = \int -\frac{1}{u} du = -lnu = -\ln(e^{-x} + 1)$

(2) $\int e^{2x}(1 + e^x)^{\frac{1}{2}} dx$

Solution

'Tidying-up' substitution:

Let $u = 1 + e^x$, so that $du = e^x dx$

$I = \int (u - 1)^2 u^{\frac{1}{2}}.\frac{1}{u-1} du$

$= \int u^{\frac{3}{2}} - u^{\frac{1}{2}} du$

$= \frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}}$

$= \frac{2}{15}(1 + e^x)^{\frac{3}{2}}(3(1 + e^x) - 5)$

$= \frac{2}{15}(1 + e^x)^{\frac{3}{2}}(3e^x - 2)$

(3) $\int \frac{1}{\sqrt{x}(1+\sqrt{x})} dx$

Solution

Note: $\frac{1}{\sqrt{x}}$ integrates to a multiple of $\sqrt{x}$ (or 1+$\sqrt{x}$)

Let $u = 1 + \sqrt{x},$ so that $du = \frac{1}{2}x^{-\frac{1}{2}}\,dx$

$I = 2\int \frac{1}{u}\,du$

$= 2\ln u = 2\ln\left(1 + \sqrt{x}\right)$

(4) $\int secx\,dx$

Solution

Method 1

Note: This is an odd power of $cosx$

$I = \int \frac{cos}{cos^2x}dx$

$= \int \frac{cosx}{1-sin^2x}dx$

Let $u = sinx$, so that $du = cosx\,dx$

$\int \frac{cosx}{1-sin^2x}dx = \int \frac{1}{1-u^2}du$

$\frac{1}{2}\int \frac{1}{1-u} + \frac{1}{1+u}du$

$= \frac{1}{2}\{-\ln|1-u| + \ln|1+u|\}$

$= \frac{1}{2}ln\left|\frac{1+sinx}{1-sinx}\right|$

$= \frac{1}{2}\ln\left|\frac{(1+sinx)^2}{(1-sinx)(1+sinx)}\right|$

$= \frac{1}{2}ln\left|\frac{(1+sinx)^2}{1-sin^2x}\right|$

$= ln\left|\frac{1+sinx}{cosx}\right|$

$= \ln|secx + tanx|$

Method 2

Let $t = tan\left(\frac{x}{2}\right)$, so that $tanx = tan\left(\frac{x}{2}+\frac{x}{2}\right) = \frac{2t}{1-t^2}$

Then it can be shown that $2t, 1-t^2$ & $1+t^2$ form the sides of a right-angled triangle, so that $secx = \frac{1+t^2}{1-t^2}$

Also, $dt = sec^2\left(\frac{x}{2}\right)\left(\frac{1}{2}\right)dx = \frac{1}{2}(1+t^2)dx$

Then $I = \int \frac{1+t^2}{1-t^2}\left(\frac{2}{1+t^2}\right)dt = 2\int \frac{1}{1-t^2}\ dt = \int \frac{1}{1-t} + \frac{1}{1+t}\ dt$

$= ln\left|\frac{1+t}{1-t}\right| = ln\left|\frac{(1+t)^2}{1-t^2}\right|$

$= \ln\left|\frac{1+t^2}{1-t^2} + \frac{2t}{1-t^2}\right| = \ln|secx + tanx|$

(5) $\int \frac{(lnx)^2}{x}dx$

Solution

Noting that $\frac{1}{x}$ integrates to lnx,

let $u = lnx$, so that $du = \frac{1}{x}dx$

$I = \int u^2du = \frac{1}{3}u^3 = \frac{1}{3}(\ln x)^3$

(6) $\int \frac{x^5}{4-x^3}dx$

Solution

Let $u = 4 - x^3$, so that $du = -3x^2 dx$

[A slightly speculative 'tidying-up' substitution.]

$I = -\frac{1}{3}\int \frac{x^3 du}{u}$

$= -\frac{1}{3}\int \frac{4-u}{u} du$

$= -\frac{1}{3}\int \frac{4}{u} - 1\, du$

$= -\frac{1}{3}(4\ln|u| - u)$

$= -\frac{1}{3}(4\ln|4 - x^3| + x^3 - 4)$

or $-\frac{1}{3}(4\ln|4 - x^3| + x^3)$, as the 4 can be included in the constant of integration

Alternative method

$\int \frac{x^5}{4-x^3} dx = \int \frac{x^5-4x^2}{4-x^3} + \frac{4x^2}{4-x^3}\; dx$

$= \int -x^2\, dx + \frac{4}{3}\int \frac{3x^2}{4-x^3}\; dx$

For the 2nd integral (J), let $u = x^3$, so that $du = 3x^2 dx$

and $J = \frac{4}{3}\int \frac{1}{4-u}\; du = -\frac{4}{3}\ln|4 - x^3|$,

so that $I = -\frac{1}{3}x^3 - \frac{4}{3}\ln|4 - x^3|$

(7) $\int \sqrt{1 + sin2x}\; dx$

Solution

$1 + sin2x = 1 + \cos\left(\frac{\pi}{2} - 2x\right) = 1 + \cos2\left(\frac{\pi}{4} - x\right) = 2cos^2(\frac{\pi}{4} - x)$

so that $\sqrt{1 + sin2x} = \sqrt{2}\cos\left(\frac{\pi}{4} - x\right)$ and I can be found easily.

(8) $\int arctanx\, dx$

Solution

Applying Parts; integrating 1:

$I = x\ arctanx - \int \frac{x}{x^2+1}\, dx = x\ arctanx - \frac{1}{2}\ln(x^2 + 1)$

(9) $\int \frac{1}{xlnx}\, dx$

Solution

$= \int \frac{\left(\frac{1}{x}\right)}{lnx}\, dx$, and as $\int \frac{1}{x}\, dx = lnx,\ let\ u = lnx, so\ that\ du = \frac{1}{x}\, dx$

Then $I = \int \frac{1}{u}\, du = \ln(lnx)$

If we try Parts, we get:

$I = lnx\ \left(\frac{1}{lnx}\right) - \int lnx(-1)(lnx)^{-2}\left(\frac{1}{x}\right) dx$

$= 1 + \int \frac{1}{xlnx}\, dx = 1 + I$

This isn't a contradiction, because we haven't allowed for the arbitrary constant. However the method doesn't give us an answer.

(10) $\int \frac{e^x}{e^{2x}+1}\,dx$

Solution

As $\int e^x\,dx = e^x$, and the rest of the integrand is a function of e^x that we can integrate, let $u = e^x$, so that $du = e^x\,dx$

Then $I = \int \frac{1}{u^2+1}\,du = \arctan(e^x)$

(11) $\int \frac{sec^2x}{4+tan^2x}\,dx$

Solution

As $\int sec^2x\,dx = tanx$, let $u = tanx$, so that $du = sec^2x\,dx$,

and $I = \int \frac{1}{4+u^2}\,du = \frac{1}{2}arctan\left(\frac{tanx}{2}\right)$

(12) $\int \frac{1}{1-sinx}\,dx$

Solution

$I = \int \frac{1+sinx}{1-sin^2x}\,dx = \int \frac{1+sinx}{cos^2x}\,dx$

$= \int sec^2x\,dx + J = tanx + J$

For J: let $u = cosx$, so that $du = -sinx\,dx$

and $J = -\int u^{-2}\,du = u^{-1} = secx$

So $I = tanx + secx$

(13) $\int_{-\frac{3}{2}}^{\frac{3}{2}} \frac{1}{4x^2+9}\ dx$

Solution

Method 1

$I = \frac{1}{4}\int_{-\frac{3}{2}}^{\frac{3}{2}} \frac{1}{x^2+\left(\frac{3}{2}\right)^2}\ dx = \frac{1}{4}\left[\frac{1}{\left(\frac{3}{2}\right)} arctan\left(\frac{x}{\left(\frac{3}{2}\right)}\right)\right]_{-\frac{3}{2}}^{\frac{3}{2}}$

$= \frac{1}{6}(\arctan(1) - \arctan(-1)) = \frac{1}{6}(\frac{\pi}{4} - \left(-\frac{\pi}{4}\right))$

$= \frac{\pi}{12}$

Method 2

Let $u = 2x$, so that $du = 2dx$ and $I = \int_{-3}^{3} \frac{1}{u^2+9}\ \left(\frac{1}{2}\right) du$

$= \frac{1}{2}\left[\left(\frac{1}{3}\right) arctan\left(\frac{u}{3}\right)\right]_{-3}^{3} = \frac{1}{6}(\arctan(1) - \arctan(-1))$

$= \frac{1}{6}(\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)) = \frac{\pi}{12}$

(14) $\int \frac{x}{1+x^4}\ dx$

Solution

As $\int x\ dx = \frac{1}{2}x^2$, and $\frac{1}{1+u^2}$ can be integrated,

let $u = x^2$, so that $du = 2x\ dx$,

and $I = \frac{1}{2}\int \frac{1}{1+u^2}\ du = \frac{1}{2}arctan(x^2)$

(15) $\int \frac{sin2x}{1+cosx}\, dx$

Solution

As a (speculative) simplifying substitution, let $u = 1 + cosx$,

so that $du = -sinx\, dx$

Then $I = 2\int \frac{sinxcosx}{u}\left(-\frac{du}{sinx}\right) = -2\int \frac{u-1}{u}\, du$

$= -2(u - lnu) = 2(\ln(1 + cosx) - cosx - 1)$

or $2(\ln(1 + cosx) - cosx)$,

including the -2 in the constant of integration

(16)* $\int_0^1 \sqrt{16x^2 + 9}\, dx$

Solution

Let $4x = 3tan\theta$, so that $4dx = 3sec^2\theta\, d\theta$

$\Rightarrow I = \int_0^{\arctan(\frac{4}{3})} \sqrt{9tan^2\theta + 9}\, \left(\frac{3}{4}\right) sec^2\theta\, d\theta$

$= \frac{9}{4}\int_0^{\arctan(\frac{4}{3})} sec^3\theta\, d\theta$

To find $J = \int sec^3\theta\, d\theta$:

Method 1 (Parts - speculative)

$J = \int sec^2\theta sec\theta\, d\theta$

Integrating $sec^2\theta$ and differentiating $sec\theta = (cos\theta)^{-1}$:

$J = tan\theta sec\theta - \int tan\theta(-1)(cos\theta)^{-2}(-sin\theta)d\theta$

$= tan\theta sec\theta - \int tan^2\theta sec\theta\, d\theta$

$= tan\theta sec\theta - \int(sec^2\theta - 1)sec\theta\, d\theta$

$= tan\theta sec\theta - J + \int sec\theta \, d\theta$

$\Rightarrow 2J = tan\theta sec\theta + \ln|sec\theta + tan\theta|$

[from the formulae booklet]

So $I = \frac{9}{8}[tan\theta sec\theta + \ln|sec\theta + tan\theta|]_0^{\arctan(\frac{4}{3})}$

$\theta = \arctan\left(\frac{4}{3}\right) \Rightarrow tan\theta = \frac{4}{3} \Rightarrow sec\theta = \frac{5}{3}$ (from a 3,4,5 triangle)

Then $I = \frac{9}{8}\{\left(\frac{4}{3}\right)\left(\frac{5}{3}\right) + \ln\left|\frac{5}{3} + \frac{4}{3}\right| - 0 - \ln|1|\}$

$= \frac{9}{8}\left\{\frac{20}{9} + ln3\right\} = \frac{5}{2} + \frac{9}{8}ln3$

Method 2

$J = \int \frac{1}{cos^3\theta} d\theta = \int \frac{cos\theta}{(1-sin^2\theta)^2} \, d\theta$

Let $u = sin\theta$, so that $du = cos\theta \, d\theta$

and $J = \int \frac{1}{(1-u^2)^2} \, du$

$\frac{1}{(1-u^2)^2} = \frac{1}{(1-u)^2(1+u)^2} = \frac{A}{1-u} + \frac{B}{(1-u)^2} + \frac{C}{1+u} + \frac{D}{(1+u)^2}$

so that $1 = A(1-u)(1+u)^2 + B(1+u)^2 + C(1+u)(1-u)^2$

$+D(1-u)^2$

Then $u = 1 \Rightarrow 1 = 4B \Rightarrow B = \frac{1}{4}$

and $u = -1 \Rightarrow 1 = 4D \Rightarrow D = \frac{1}{4}$

$u = 0 \Rightarrow 1 = A + B + C + D$

Equating coeffs of u^3: $0 = -A + C$

Hence $1 = A + \frac{1}{4} + A + \frac{1}{4}$ and so $A = C = \frac{1}{4}$

Therefore $J = \frac{1}{4}\int \frac{1}{1-u} + \frac{1}{(1-u)^2} + \frac{1}{1+u} + \frac{1}{(1+u)^2}\ du$

$= \frac{1}{4}(-\ln|1-u| + (1-u)^{-1} + \ln|1+u| - (1+u)^{-1})$

$= \frac{1}{4}(\ln\left|\frac{1+u}{1-u}\right| + \frac{(1+u)-(1-u)}{(1-u)(1+u)})$

$= \frac{1}{4}(\ln\left|\frac{1+u}{1-u}\right| + \frac{2u}{1-u^2})$

$= \frac{1}{4}(ln\left|\frac{1+sin\theta}{1-sin\theta}\right| + \frac{2sin\theta}{cos^2\theta})$

$= \frac{1}{4}(ln\left|\frac{(1+sin\theta)^2}{(1-sin\theta)(1+sin\theta)}\right| + 2tan\theta sec\theta)$

$= \frac{1}{4}(ln\left|\frac{(1+sin\theta)^2}{cos^2\theta}\right| + 2tan\theta sec\theta)$

$= \frac{1}{2}(ln\left|\frac{1+sin\theta}{cos\theta}\right| + tan\theta sec\theta)$

$= \frac{1}{2}(ln|sec\theta + tan\theta| + tan\theta sec\theta)$

(then as before)

(17) $\int sec^3x\ dx$

Solution

See (16)

(18) $\int sec^4x\ dx$

Solution

$I = \int sec^2x(1 + tan^2x)\ dx = tanx + \int sec^2x tan^2x\ dx$

Then let $u = tanx$, so that $du = sec^2x\ dx$, and $I = tanx + \int u^2 \, du = tanx + \frac{1}{3} tan^3 x$

(19) $\int cos^5 x \, dx$

Solution

$I = \int cosx(1 - sin^2 x)^2 \, dx$

$= \int cosx \, dx - 2 \int cosx \, sin^2 x \, dx + \int cosx \, sin^4 x \, dx$

Let $u = sinx$, so that $du = cosx \, dx$

Then $I = sinx - 2 \int u^2 \, du + \int u^4 \, du$

$= sinx - \frac{2}{3} sin^3 x + \frac{1}{5} sin^5 x$

(20) $\int cosx \ln(cosx) \, dx$

Solution

By Parts, integrating $cosx$ & differentiating $ln(cosx)$:

$I = sinxln(cosx) - \int sinx \left(\frac{1}{cosx}\right)(-sinx)dx$

$= sinxln(cosx) + \int (1 - cos^2 x) secx \, dx$

$= sinxln(cosx) + J,$

where $J = \int secx \, dx - \int cosx \, dx$

$= \ln|secx + tanx| - sinx$

So $I = sinxln(cosx) + ln|secx + tanx| - sinx$

(21) $\int tanx\, sinx\, dx$

Solution

$I = \int \frac{sin^2x}{cosx}\, dx = \int (1 - cos^2x) secx\, dx$

$= \int secx\, dx - \int cosx\, dx = ln|secx + tanx| - sinx$

(22) $\int sin3xcosx\, dx$

Solution

Method 1

$sin4x = sin(3x + x) = sin3xcosx + cos3xsinx$

& $sin2x = sin(3x - x) = sin3xcosx - cos3xsinx$,

so that $sin3xcosx = \frac{1}{2}(sin4x + sin2x)$

and $I = \frac{1}{2}\int sin4x + sin2x\, dx = \frac{1}{2}(-\frac{1}{4}cos4x - \frac{1}{2}cos2x)$

$= -\frac{1}{8}(cos4x + 2cos2x)$

Method 2

$sin3x = sin(2x + x) = sin2xcosx + cos2xsinx$

$= 2sinxcos^2x + (cos^2x - sin^2x)sinx$

$= 3sinxcos^2x - sin^3x$

so that $I = \int 3sinxcos^3x\, dx - \int sin^3xcosx\, dx$

Let $u = cosx$ for the 1st integral, so that $du = -sinx\, dx$,

and let $v = sinx$ for the 2nd integral, so that $dv = cosx\, dx$

Then $I = -3\int u^3 du - \int v^3 dv$

$= -\frac{3}{4}cos^4x - \frac{1}{4}sin^4x$

$= -\frac{1}{4}(3cos^4x + sin^4x)$

[As a check, we can substitute values of x such as 0 & $\frac{\pi}{2}$, to compare the answers by the two methods. There is in fact a difference of $\frac{3}{8}$ in each case, but this is due to the constant of integration.]

Method 3

By Parts, integrating $cosx$ and differentiating $sin3x$ (for example):

$I = sinxsin3x - \int sinx(3cos3x)dx$

Then integrating $sinx$ and differentiating $cos3x$,

$I = sinxsin3x - 3\{-cosxcos3x - \int(-cosx)(-3sin3x)dx\}$

$= sinxsin3x + 3cosxcos3x + 9I$

and hence $I = -\frac{1}{8}(sinxsin3x + 3cosxcos3x)$

[Substituting values of x shows agreement with Method 1.]

(23) $\int cosecx\, dx$

Solution

Method 1

$I = \int \frac{1}{sinx}\ dx = \int \frac{sinx}{sin^2x}\ dx = \int \frac{sinx}{1-cos^2x}\ dx$

Let $u = cosx$, so that $du = -sinx\ dx$

Then $I = -\int \frac{1}{1-u^2} du = -\frac{1}{2}\int \frac{1}{1-u} + \frac{1}{1+u}\ du$

$= -\frac{1}{2}(ln|1+u| - ln|1-u|)$

$= -\frac{1}{2} ln|\frac{1+cosx}{1-cosx}|$

Method 2

Let $t = tan\left(\frac{x}{2}\right)$, so that $tanx = tan\left(\frac{x}{2}+\frac{x}{2}\right) = \frac{2t}{1-t^2}$

Then it can be shown that $2t, 1-t^2$ & $1+t^2$ form the sides of a right-angled triangle, so that $cosecx = \frac{1+t^2}{2t}$

Also, $dt = sec^2\left(\frac{x}{2}\right)\left(\frac{1}{2}\right)dx = \frac{1}{2}(1+t^2)dx$

so that $I = \int \frac{1+t^2}{2t}\left(\frac{2}{1+t^2}\right)dt = \int \frac{1}{t}dt = ln|\,tan\left(\frac{x}{2}\right)|$

[To reconcile these two answers:

rtp (result to prove): $ln\left|\frac{1-cosx}{1+cosx}\right| = 2ln|tan\left(\frac{x}{2}\right)|$

$\Leftrightarrow \frac{1-cosx}{1+cosx} = tan^2\left(\frac{x}{2}\right)$

$LHS = \frac{1-\left(\frac{1-t^2}{1+t^2}\right)}{1+\left(\frac{1-t^2}{1+t^2}\right)} = \frac{(1+t^2)-(1-t^2)}{(1+t^2)+(1-t^2)} = t^2 = RHS$]

(24) $\int tan^3x\,dx$

Solution

$I = \int(sec^2x - 1)tanx\,dx = \int sec^2xtanx\,dx - \int tanx\,dx$

For the 1st integral, $\int sec^2x\,dx = tanx$, so let $u = tanx$

Then $du = sec^2x\,dx$ and $I = \int u\,du - ln|secx|$

$= \frac{1}{2}tan^2x - ln|secx|$

(25) $\int tan^4x\,dx$

Solution

$I = \int(sec^2x - 1)\,tan^2x\,dx$

For the 1st integral, $\int sec^2x\,dx = tanx$, so let $u = tanx$

Then $du = sec^2x\,dx$ and $I = \int u^2\,du - \int sec^2x - 1\,dx$

$= \frac{1}{3}tan^3x - tanx + x$

(26) $\int \frac{cos^3x}{sin^2x}\,dx$

Solution

$I = \int \frac{cosx(1-sin^2x)}{sin^2x}\,dx$

Let $u = sinx$, so that $du = cosx\,dx$,

and $I = \int \frac{1-u^2}{u^2}\,du = \int u^{-2} - 1\,du = -u^{-1} - u = -cosecx - sinx$

(27) $\int \frac{cosx+sinx}{cosx-sinx}\ dx$

Solution

As $\int cosx + sinx\ dx = sinx - cosx$,

let $u = sinx - cosx$, so that $du = cosx + sinx\ dx$,

and $I = -\int \frac{1}{u}\ du = -ln|sinx - cosx|$

(28) $\int tanx\ secx\ dx$

Solution

$I = \int \frac{sinx}{cos^2x}\ dx$

Let $u = cosx$, so that $du = -sinx\ dx$,

and $I = -\int u^{-2}\ du = u^{-1} = secx$

(29) $\int arcsin\left(\frac{x}{3}\right)\ dx$

Solution

By Parts, integrating 1 and differentiating $arcsin\left(\frac{x}{3}\right)$:

$I = xarcsin\left(\frac{x}{3}\right) - \int x\left(\frac{1}{\sqrt{1-\left(\frac{x}{3}\right)^2}}\right)\left(\frac{1}{3}\right) dx$

$= xarcsin\left(\frac{x}{3}\right) - \int \frac{x}{\sqrt{9-x^2}}\ dx$

Let $u = x^2$, so that $du = 2x\ dx$

Then $I = xarcsin\left(\frac{x}{3}\right) - \frac{1}{2}\int \frac{1}{\sqrt{9-u}}\ du$

$= xarcsin\left(\frac{x}{3}\right) - \frac{1}{2}(9-u)^{\frac{1}{2}}\left(\frac{1}{\frac{1}{2}}\right)(-1)$

$= xarcsin\left(\frac{x}{3}\right) + (9-x^2)^{\frac{1}{2}}$

(30)* $\int \frac{sinx}{sinx+cosx}\ dx$

Solution

Note that $\int \frac{sinx-cosx}{sinx+cosx}\ dx = -\ln|sinx+cosx|$,

as $\frac{d}{dx}(sinx+cosx) = cosx - sinx$

Let $sinx = A(sinx+cosx) + B(sinx-cosx)$

Then $A + B = 1$ & $A - B = 0$, so $A = B = \frac{1}{2}$

and $I = \frac{1}{2}\int \frac{sinx+cosx}{sinx+cosx} + \frac{sinx-cosx}{sinx+cosx}\ dx$

$= \frac{1}{2}x - \frac{1}{2}ln|sinx+cosx|$

(31) Find a reduction formula for $I_n = \int_0^1 x^n\sqrt{1-x^2}\ dx$

Solution

By Parts, integrating $x\sqrt{1-x^2}$ and differentiating x^{n-1},

$I_n = \left[-\frac{1}{2}\frac{(1-x^2)^{\frac{3}{2}}}{\left(\frac{3}{2}\right)}x^{n-1}\right]_0^1 - \int_0^1 -\frac{1}{2}\frac{(1-x^2)^{\frac{3}{2}}}{\left(\frac{3}{2}\right)}(n-1)x^{n-2}dx$

$= -\frac{1}{3}\left[(1-x^2)^{\frac{3}{2}}x^{n-1}\right]_0^1 + \frac{1}{3}(n-1)\int_0^1 (1-x^2)^{\frac{3}{2}}x^{n-2}dx$

$= -\frac{1}{3}(0-0) + \frac{1}{3}(n-1)\int_0^1 (1-x^2)\sqrt{1-x^2}\ x^{n-2}dx$

$= \frac{1}{3}(n-1)\{I_{n-2} - I_n\}$

so that $I_n\big(3 + (n-1)\big) = (n-1)I_{n-2}$

and hence $I_n = \frac{n-1}{n+2} I_{n-2}$

(32) $\int_1^e (lnx)^2\, dx$

Solution

By Parts, integrating 1 and differentiating $(lnx)^2$:

$I = [x(lnx)^2]_1^e - \int_1^e x(2lnx)\left(\frac{1}{x}\right)\, dx$

$= e - 2\int_1^e lnx\, dx$

Then, by Parts again, integrating 1 and differentiating lnx:

$I = e - 2\{[xlnx]_1^e - \int_1^e x\left(\frac{1}{x}\right)\, dx\}$

$= e - 2e + 2(e-1) = e - 2$

(33) $\int 2^x\, dx$

Solution

Let $2 = e^k$, so that $I = \int e^{kx}\, dx = \frac{1}{k}e^{kx} = \frac{1}{ln2}2^x$

(34) $\int \frac{1}{x^2+6x+18}\ dx$

Solution

$I = \int \frac{1}{(x+3)^2+9}\ dx = \frac{1}{3} arctan\left(\frac{x+3}{3}\right)$

(35) $\int \frac{x^2}{1+x^6}\ dx$

Solution

Let $u = x^3$, so that $du = 3x^2 dx$,

and $I = \frac{1}{3}\int \frac{1}{1+u^2}\ du = \frac{1}{3} arctanu = \frac{1}{3} arctan(x^3)$

(36) $\int \frac{1}{(2x^2+3)^{\frac{3}{2}}}\ dx$

Solution

Let $2x^2 = 3tan^2\theta$, so that $x = \sqrt{\frac{3}{2}}\ tan\theta$ and $dx = \sqrt{\frac{3}{2}}\ sec^2\theta\ d\theta$

Then $I = \sqrt{\frac{3}{2}}\int \frac{sec^2\theta}{3^{\frac{3}{2}}(tan^2\theta+1)^{\frac{3}{2}}}\ d\theta$

$= \frac{1}{3\sqrt{2}}\int \frac{sec^2\theta}{sec^3\theta}\ d\theta = \frac{1}{3\sqrt{2}}\int cos\theta\ d\theta = \frac{1}{3\sqrt{2}} sin\theta$

Now $sin^2\theta = 1 - cos^2\theta = 1 - \frac{1}{1+tan^2\theta} = 1 - \frac{1}{1+\frac{2}{3}x^2}$

$= 1 - \frac{3}{3+2x^2} = \frac{2x^2}{3+2x^2}$

so that $I = \frac{1}{3\sqrt{2}}\sqrt{\frac{2x^2}{3+2x^2}} = \frac{x}{3\sqrt{3+2x^2}}$

(37)** $\int \sqrt{4x^2 - 1}\, dx$

Solution

Let $2x = sec\theta$, so that $2dx = tan\theta sec\theta\; d\theta$

Then $I = \int \sqrt{sec^2\theta - 1}\, (\frac{1}{2}) tan\theta sec\theta\; d\theta$

$= \frac{1}{2}\int tan^2\theta sec\theta\; d\theta$

$= \frac{1}{2}\int (sec^2\theta - 1) sec\theta\; d\theta$

$= \frac{1}{2}\int sec^3\theta\; d\theta - \frac{1}{2}\int sec\theta\; d\theta$

Then see (4) for $\int sec\theta\; d\theta$ and (17) for $\int sec^3\theta\; d\theta$.

Alternative method: See Integration Exercises - Part 3 (Hyperbolic Functions).

(38) $\int \frac{4x+5}{\sqrt{4-6x-x^2}}\, dx$

Solution

$I = -2\int \frac{-6-2x}{\sqrt{4-6x-x^2}}\, dx - \int \frac{7}{\sqrt{4-6x-x^2}}\, dx$

$= -\frac{2(4-6x-x^2)^{\frac{1}{2}}}{\left(\frac{1}{2}\right)} - 7\int \frac{1}{\sqrt{13-(x+3)^2}}\, dx$

$= -4(4 - 6x - x^2)^{\frac{1}{2}} - 7 arcsin(\frac{x+3}{\sqrt{13}})$

(39) $\int \frac{sin\sqrt{x}}{\sqrt{x}}\ dx$

Solution

Let $y = \sqrt{x}$, so that $dy = \frac{1}{2}x^{-\frac{1}{2}}\, dx$

and $I = \int siny\ .\ 2\ dy = -2cos\sqrt{x}$

(40) Find a reduction formula for $I_n = \int_0^2 x^n\sqrt{4 - x^2}\ \ dx$, and hence show that $\int_0^2 x^5\sqrt{4 - x^2}\ \ dx = \frac{1024}{105}$

Solution

By Parts, integrating $x\sqrt{4 - x^2}$ and differentiating x^{n-1},

$$I_n = \left[-\frac{1}{2}\frac{(4-x^2)^{\frac{3}{2}}}{\left(\frac{3}{2}\right)}x^{n-1}\right]_0^2 - \int_0^2 -\frac{1}{2}\frac{(4-x^2)^{\frac{3}{2}}}{\left(\frac{3}{2}\right)}(n-1)x^{n-2}dx$$

$$= -\frac{1}{3}\left[(4 - x^2)^{\frac{3}{2}}x^{n-1}\right]_0^2 + \frac{1}{3}(n - 1)\int_0^2 (4 - x^2)^{\frac{3}{2}}\, x^{n-2}dx$$

$$= -\frac{1}{3}(0 - 0) + \frac{1}{3}(n - 1)\int_0^2 (4 - x^2)\sqrt{4 - x^2}\ x^{n-2}dx$$

$$= \frac{1}{3}(n - 1)\{4I_{n-2} - I_n\}$$

so that $I_n\big(3 + (n - 1)\big) = 4(n - 1)I_{n-2}$

and hence $I_n = \frac{4(n-1)}{n+2}I_{n-2}$

Then $\int_0^2 x^5\sqrt{4 - x^2}\ \ dx = I_5 = \frac{4(4)}{7}I_3 = \frac{16}{7}\frac{(4)(2)}{5}I_1 = \frac{128}{35}I_1$

and $I_1 = \int_0^2 x\sqrt{4 - x^2}\ dx = -\frac{1}{2}\int_0^2 -2x\sqrt{4 - x^2}\ dx$

$$= -\frac{1}{2}\left[\frac{(4-x^2)^{\frac{3}{2}}}{\left(\frac{3}{2}\right)}\right]_0^2 = -\frac{1}{3}(0 - 8) = \frac{8}{3},$$

so that $\int_0^2 \boldsymbol{x^5\sqrt{4-x^2}\;dx} = \frac{\mathbf{128}}{\mathbf{35}}\left(\frac{\mathbf{8}}{\mathbf{3}}\right) = \frac{\mathbf{1024}}{\mathbf{105}}$

(41) Find a reduction formula for $I_n = \int_0^{\pi} cos^n x\, dx$, and hence show that $\int_0^{\pi} cos^4 x \;dx = \frac{3\pi}{8}$

Solution

$I_n = \int_0^{\pi} cosx\,.\; cos^{n-1}x\, dx$

Integrating by Parts (integrating $cosx$ and differentiating $cos^{n-1}x$),

$I_n = [sinx. cos^{n-1}x]_0^{\pi} - \int_0^{\pi} sinx(n-1)cos^{n-2}(-sinx)dx$

$= 0 + (n-1)\int_0^{\pi}(1-cos^2x)\; cos^{n-2}dx$

$= (n-1)I_{n-2} - (n-1)I_n$,

so that $nI_n = (n-1)I_{n-2}$

and $I_n = \frac{n-1}{n} I_{n-2}$

Hence $\int_0^{\pi} cos^4 x = \frac{3}{4} I_2 = \frac{3}{4} \cdot \frac{1}{2} I_0 = \frac{3}{8}\int_0^{\pi} 1\, dx = \frac{3}{8}[x]_0^{\pi} = \frac{3}{8}(\pi - 0) = \frac{3\pi}{8}$