

Integration Exercises - Part 2 (Sol'ns)(27 pages; 4/7/26)

(Note: The constant of integration has been omitted throughout.)

$$(1) \int \frac{1}{1+e^x} dx$$

Solution

Step 1: rearrange

$$I = \int \frac{e^{-x}}{e^{-x}+1} dx$$

Step 2: substitute $u = e^{-x} + 1$, so that $du = -e^{-x}dx$

$$I = \int -\frac{1}{u} du = -\ln u = -\ln(e^{-x} + 1)$$

$$(2) \int e^{2x}(1 + e^x)^{\frac{1}{2}} dx$$

Solution

'Tidying-up' substitution:

Let $u = 1 + e^x$, so that $du = e^x dx$

$$I = \int (u - 1)^2 u^{\frac{1}{2}} \cdot \frac{1}{u-1} du$$

$$= \int u^{\frac{3}{2}} - u^{\frac{1}{2}} du$$

$$= \frac{2}{5} u^{\frac{5}{2}} - \frac{2}{3} u^{\frac{3}{2}}$$

$$= \frac{2}{15} (1 + e^x)^{\frac{3}{2}} (3(1 + e^x) - 5)$$

$$= \frac{2}{15} (1 + e^x)^{\frac{3}{2}} (3e^x - 2)$$

$$(3) \int \frac{1}{\sqrt{x}(1+\sqrt{x})} dx$$

Solution

Note: $\frac{1}{\sqrt{x}}$ integrates to a multiple of $\sqrt{x}$ (or $1+\sqrt{x}$)

Let $u = 1 + \sqrt{x}$, so that $du = \frac{1}{2}x^{-\frac{1}{2}} dx$

$$I = 2 \int \frac{1}{u} du$$

$$= 2 \ln u = 2 \ln(1 + \sqrt{x})$$

$$(4) \int \sec x \, dx$$

Solution**Method 1**

Note: This is an odd power of $\cos x$

$$I = \int \frac{\cos}{\cos^2 x} dx$$

$$= \int \frac{\cos x}{1 - \sin^2 x} dx$$

Let $u = \sin x$, so that $du = \cos x \, dx$

$$\int \frac{\cos x}{1 - \sin^2 x} dx = \int \frac{1}{1 - u^2} du$$

$$\frac{1}{2} \int \frac{1}{1 - u} + \frac{1}{1 + u} du$$

$$= \frac{1}{2} \{-\ln |1 - u| + \ln |1 + u|\}$$

$$= \frac{1}{2} \ln \left| \frac{1 + \sin x}{1 - \sin x} \right|$$

$$= \frac{1}{2} \ln \left| \frac{(1 + \sin x)^2}{(1 - \sin x)(1 + \sin x)} \right|$$

$$= \frac{1}{2} \ln \left| \frac{(1 + \sin x)^2}{1 - \sin^2 x} \right|$$

$$= \ln \left| \frac{1+\sin x}{\cos x} \right|$$

$$= \ln |\sec x + \tan x|$$

Method 2

Let $t = \tan\left(\frac{x}{2}\right)$, so that $\tan x = \tan\left(\frac{x}{2} + \frac{x}{2}\right) = \frac{2t}{1-t^2}$

Then it can be shown that $2t$, $1-t^2$ & $1+t^2$ form the sides of a right-angled triangle, so that $\sec x = \frac{1+t^2}{1-t^2}$

Also, $dt = \sec^2\left(\frac{x}{2}\right)\left(\frac{1}{2}\right)dx = \frac{1}{2}(1+t^2)dx$

Then $I = \int \frac{1+t^2}{1-t^2} \left(\frac{2}{1+t^2}\right) dt = 2 \int \frac{1}{1-t^2} dt = \int \frac{1}{1-t} + \frac{1}{1+t} dt$

$$= \ln \left| \frac{1+t}{1-t} \right| = \ln \left| \frac{(1+t)^2}{1-t^2} \right|$$

$$= \ln \left| \frac{1+t^2}{1-t^2} + \frac{2t}{1-t^2} \right| = \ln |\sec x + \tan x|$$

(5) $\int \frac{(\ln x)^2}{x} dx$

Solution

Noting that $\frac{1}{x}$ integrates to $\ln x$,

let $u = \ln x$, so that $du = \frac{1}{x} dx$

$$I = \int u^2 du = \frac{1}{3} u^3 = \frac{1}{3} (\ln x)^3$$

(6) $\int \frac{x^5}{4-x^3} dx$

Solution

Let $u = 4 - x^3$, so that $du = -3x^2 dx$

[A slightly speculative 'tidying-up' substitution.]

$$\begin{aligned}
 I &= -\frac{1}{3} \int \frac{x^3 du}{u} \\
 &= -\frac{1}{3} \int \frac{4-u}{u} du \\
 &= -\frac{1}{3} \int \frac{4}{u} - 1 du \\
 &= -\frac{1}{3} (4 \ln |u| - u) \\
 &= -\frac{1}{3} (4 \ln |4 - x^3| + x^3 - 4)
 \end{aligned}$$

or $-\frac{1}{3} (4 \ln |4 - x^3| + x^3)$, as the 4 can be included in the constant of integration

Alternative method

$$\begin{aligned}
 \int \frac{x^5}{4-x^3} dx &= \int \frac{x^5-4x^2}{4-x^3} + \frac{4x^2}{4-x^3} dx \\
 &= \int -x^2 dx + \frac{4}{3} \int \frac{3x^2}{4-x^3} dx
 \end{aligned}$$

For the 2nd integral (J), let $u = x^3$, so that $du = 3x^2 dx$

$$\text{and } J = \frac{4}{3} \int \frac{1}{4-u} du = -\frac{4}{3} \ln |4 - x^3|,$$

$$\text{so that } I = -\frac{1}{3} x^3 - \frac{4}{3} \ln |4 - x^3|$$

$$(7) \int \sqrt{1 + \sin 2x} dx$$

Solution

$$1 + \sin 2x = 1 + \cos\left(\frac{\pi}{2} - 2x\right) = 1 + \cos 2\left(\frac{\pi}{4} - x\right) = 2\cos^2\left(\frac{\pi}{4} - x\right)$$

so that $\sqrt{1 + \sin 2x} = \sqrt{2} \cos\left(\frac{\pi}{4} - x\right)$ and I can be found easily.

$$(8) \int \arctan x \, dx$$

Solution

Applying Parts; integrating 1:

$$I = x \arctan x - \int \frac{x}{x^2+1} \, dx = x \arctan x - \frac{1}{2} \ln(x^2 + 1)$$

$$(9) \int \frac{1}{x \ln x} \, dx$$

Solution

$$= \int \frac{\left(\frac{1}{x}\right)}{\ln x} \, dx, \text{ and as } \int \frac{1}{x} \, dx = \ln x, \text{ let } u = \ln x, \text{ so that } du = \frac{1}{x} \, dx$$

$$\text{Then } I = \int \frac{1}{u} \, du = \ln(\ln x)$$

If we try Parts, we get:

$$I = \ln x \left(\frac{1}{\ln x}\right) - \int \ln x (-1)(\ln x)^{-2} \left(\frac{1}{x}\right) \, dx$$

$$= 1 + \int \frac{1}{x \ln x} \, dx = 1 + I$$

This isn't a contradiction, because we haven't allowed for the arbitrary constant. However the method doesn't give us an answer.

$$(10) \int \frac{e^x}{e^{2x}+1} dx$$

Solution

As $\int e^x dx = e^x$, and the rest of the integrand is a function of e^x that we can integrate, let $u = e^x$, so that $du = e^x dx$

$$\text{Then } I = \int \frac{1}{u^2+1} du = \arctan(e^x)$$

$$(11) \int \frac{\sec^2 x}{4+\tan^2 x} dx$$

Solution

As $\int \sec^2 x dx = \tan x$, let $u = \tan x$, so that $du = \sec^2 x dx$,

$$\text{and } I = \int \frac{1}{4+u^2} du = \frac{1}{2} \arctan\left(\frac{\tan x}{2}\right)$$

$$(12) \int \frac{1}{1-\sin x} dx$$

Solution

$$I = \int \frac{1+\sin x}{1-\sin^2 x} dx = \int \frac{1+\sin x}{\cos^2 x} dx$$

$$= \int \sec^2 x dx + J = \tan x + J$$

For J : let $u = \cos x$, so that $du = -\sin x dx$

$$\text{and } J = -\int u^{-2} du = u^{-1} = \sec x$$

$$\text{So } I = \tan x + \sec x$$

$$(13) \int_{-\frac{3}{2}}^{\frac{3}{2}} \frac{1}{4x^2+9} dx$$

Solution

Method 1

$$\begin{aligned} I &= \frac{1}{4} \int_{-\frac{3}{2}}^{\frac{3}{2}} \frac{1}{x^2 + \left(\frac{3}{2}\right)^2} dx = \frac{1}{4} \left[\frac{1}{\left(\frac{3}{2}\right)} \arctan \left(\frac{x}{\left(\frac{3}{2}\right)} \right) \right]_{-\frac{3}{2}}^{\frac{3}{2}} \\ &= \frac{1}{6} (\arctan(1) - \arctan(-1)) = \frac{1}{6} \left(\frac{\pi}{4} - \left(-\frac{\pi}{4} \right) \right) \\ &= \frac{\pi}{12} \end{aligned}$$

Method 2

$$\begin{aligned} \text{Let } u &= 2x, \text{ so that } du = 2dx \text{ and } I = \int_{-3}^3 \frac{1}{u^2+9} \left(\frac{1}{2} \right) du \\ &= \frac{1}{2} \left[\left(\frac{1}{3} \right) \arctan \left(\frac{u}{3} \right) \right]_{-3}^3 = \frac{1}{6} (\arctan(1) - \arctan(-1)) \\ &= \frac{1}{6} \left(\frac{\pi}{4} - \left(-\frac{\pi}{4} \right) \right) = \frac{\pi}{12} \end{aligned}$$

$$(14) \int \frac{x}{1+x^4} dx$$

Solution

As $\int x dx = \frac{1}{2} x^2$, and $\frac{1}{1+u^2}$ can be integrated,

let $u = x^2$, so that $du = 2x dx$,

$$\text{and } I = \frac{1}{2} \int \frac{1}{1+u^2} du = \frac{1}{2} \arctan(x^2)$$

$$(15) \int \frac{\sin 2x}{1+\cos x} dx$$

Solution

As a (speculative) simplifying substitution, let $u = 1 + \cos x$,

so that $du = -\sin x dx$

$$\text{Then } I = 2 \int \frac{\sin x \cos x}{u} \left(-\frac{du}{\sin x} \right) = -2 \int \frac{u-1}{u} du$$

$$= -2(u - \ln u) = 2(\ln(1 + \cos x) - \cos x - 1)$$

$$\text{or } 2(\ln(1 + \cos x) - \cos x),$$

including the -2 in the constant of integration

$$(16)^* \int_0^1 \sqrt{16x^2 + 9} dx$$

Solution

Let $4x = 3\tan\theta$, so that $4dx = 3\sec^2\theta d\theta$

$$\Rightarrow I = \int_0^{\arctan(\frac{4}{3})} \sqrt{9\tan^2\theta + 9} \left(\frac{3}{4}\right) \sec^2\theta d\theta$$

$$= \frac{9}{4} \int_0^{\arctan(\frac{4}{3})} \sec^3\theta d\theta$$

To find $J = \int \sec^3\theta d\theta$:

Method 1 (Parts - speculative)

$$J = \int \sec^2\theta \sec\theta d\theta$$

Integrating $\sec^2\theta$ and differentiating $\sec\theta = (\cos\theta)^{-1}$:

$$J = \tan\theta \sec\theta - \int \tan\theta (-1)(\cos\theta)^{-2} (-\sin\theta) d\theta$$

$$= \tan\theta \sec\theta - \int \tan^2\theta \sec\theta d\theta$$

$$= \tan\theta \sec\theta - \int (\sec^2\theta - 1) \sec\theta d\theta$$

$$= \tan\theta \sec\theta - J + \int \sec\theta \, d\theta$$

$$\Rightarrow 2J = \tan\theta \sec\theta + \ln |\sec\theta + \tan\theta|$$

[from the formulae booklet]

$$\text{So } I = \frac{9}{8} [\tan\theta \sec\theta + \ln |\sec\theta + \tan\theta|]_{\arctan(\frac{4}{3})}^0$$

$$\theta = \arctan\left(\frac{4}{3}\right) \Rightarrow \tan\theta = \frac{4}{3} \Rightarrow \sec\theta = \frac{5}{3} \text{ (from a 3,4,5 triangle)}$$

$$\text{Then } I = \frac{9}{8} \left\{ \left(\frac{4}{3}\right) \left(\frac{5}{3}\right) + \ln \left| \frac{5}{3} + \frac{4}{3} \right| - 0 - \ln|1| \right\}$$

$$= \frac{9}{8} \left\{ \frac{20}{9} + \ln 3 \right\} = \frac{5}{2} + \frac{9}{8} \ln 3$$

Method 2

$$J = \int \frac{1}{\cos^3\theta} d\theta = \int \frac{\cos\theta}{(1-\sin^2\theta)^2} d\theta$$

Let $u = \sin\theta$, so that $du = \cos\theta \, d\theta$

$$\text{and } J = \int \frac{1}{(1-u^2)^2} du$$

$$\frac{1}{(1-u^2)^2} = \frac{1}{(1-u)^2(1+u)^2} = \frac{A}{1-u} + \frac{B}{(1-u)^2} + \frac{C}{1+u} + \frac{D}{(1+u)^2}$$

$$\text{so that } 1 = A(1-u)(1+u)^2 + B(1+u)^2 + C(1+u)(1-u)^2 + D(1-u)^2$$

$$\text{Then } u = 1 \Rightarrow 1 = 4B \Rightarrow B = \frac{1}{4}$$

$$\text{and } u = -1 \Rightarrow 1 = 4D \Rightarrow D = \frac{1}{4}$$

$$u = 0 \Rightarrow 1 = A + B + C + D$$

$$\text{Equating coeffs of } u^3: 0 = -A + C$$

$$\text{Hence } 1 = A + \frac{1}{4} + A + \frac{1}{4} \text{ and so } A = C = \frac{1}{4}$$

$$\begin{aligned}
\text{Therefore } J &= \frac{1}{4} \int \frac{1}{1-u} + \frac{1}{(1-u)^2} + \frac{1}{1+u} + \frac{1}{(1+u)^2} du \\
&= \frac{1}{4} (-\ln|1-u| + (1-u)^{-1} + \ln|1+u| - (1+u)^{-1}) \\
&= \frac{1}{4} \left(\ln \left| \frac{1+u}{1-u} \right| + \frac{(1+u)-(1-u)}{(1-u)(1+u)} \right) \\
&= \frac{1}{4} \left(\ln \left| \frac{1+u}{1-u} \right| + \frac{2u}{1-u^2} \right) \\
&= \frac{1}{4} \left(\ln \left| \frac{1+\sin\theta}{1-\sin\theta} \right| + \frac{2\sin\theta}{\cos^2\theta} \right) \\
&= \frac{1}{4} \left(\ln \left| \frac{(1+\sin\theta)^2}{(1-\sin\theta)(1+\sin\theta)} \right| + 2\tan\theta\sec\theta \right) \\
&= \frac{1}{4} \left(\ln \left| \frac{(1+\sin\theta)^2}{\cos^2\theta} \right| + 2\tan\theta\sec\theta \right) \\
&= \frac{1}{2} \left(\ln \left| \frac{1+\sin\theta}{\cos\theta} \right| + \tan\theta\sec\theta \right) \\
&= \frac{1}{2} (\ln|\sec\theta + \tan\theta| + \tan\theta\sec\theta)
\end{aligned}$$

(then as before)

$$(17) \int \sec^3 x \, dx$$

Solution

See (16)

$$(18) \int \sec^4 x \, dx$$

Solution

$$I = \int \sec^2 x (1 + \tan^2 x) \, dx = \tan x + \int \sec^2 x \tan^2 x \, dx$$

Then let $u = \tan x$, so that $du = \sec^2 x \, dx$,

$$\text{and } I = \tan x + \int u^2 du = \tan x + \frac{1}{3} \tan^3 x$$

$$(19) \int \cos^5 x \, dx$$

Solution

$$I = \int \cos x (1 - \sin^2 x)^2 \, dx$$

$$= \int \cos x \, dx - 2 \int \cos x \sin^2 x \, dx + \int \cos x \sin^4 x \, dx$$

Let $u = \sin x$, so that $du = \cos x \, dx$

$$\text{Then } I = \sin x - 2 \int u^2 \, du + \int u^4 \, du$$

$$= \sin x - \frac{2}{3} \sin^3 x + \frac{1}{5} \sin^5 x$$

$$(20) \int \cos x \ln(\cos x) \, dx$$

Solution

By Parts, integrating $\cos x$ & differentiating $\ln(\cos x)$:

$$I = \sin x \ln(\cos x) - \int \sin x \left(\frac{1}{\cos x} \right) (-\sin x) \, dx$$

$$= \sin x \ln(\cos x) + \int (1 - \cos^2 x) \sec x \, dx$$

$$= \sin x \ln(\cos x) + J,$$

$$\text{where } J = \int \sec x \, dx - \int \cos x \, dx$$

$$= \ln|\sec x + \tan x| - \sin x$$

$$\text{So } I = \sin x \ln(\cos x) + \ln|\sec x + \tan x| - \sin x$$

$$(21) \int \tan x \sin x \, dx$$

Solution

$$\begin{aligned} I &= \int \frac{\sin^2 x}{\cos x} \, dx = \int (1 - \cos^2 x) \sec x \, dx \\ &= \int \sec x \, dx - \int \cos x \, dx = \ln|\sec x + \tan x| - \sin x \end{aligned}$$

$$(22) \int \sin 3x \cos x \, dx$$

Solution

Method 1

$$\sin 4x = \sin(3x + x) = \sin 3x \cos x + \cos 3x \sin x$$

$$\& \sin 2x = \sin(3x - x) = \sin 3x \cos x - \cos 3x \sin x,$$

$$\text{so that } \sin 3x \cos x = \frac{1}{2}(\sin 4x + \sin 2x)$$

$$\begin{aligned} \text{and } I &= \frac{1}{2} \int \sin 4x + \sin 2x \, dx = \frac{1}{2} \left(-\frac{1}{4} \cos 4x - \frac{1}{2} \cos 2x \right) \\ &= -\frac{1}{8} (\cos 4x + 2 \cos 2x) \end{aligned}$$

Method 2

$$\sin 3x = \sin(2x + x) = \sin 2x \cos x + \cos 2x \sin x$$

$$= 2 \sin x \cos^2 x + (\cos^2 x - \sin^2 x) \sin x$$

$$= 3 \sin x \cos^2 x - \sin^3 x$$

$$\text{so that } I = \int 3 \sin x \cos^3 x \, dx - \int \sin^3 x \cos x \, dx$$

$$\text{Let } u = \cos x \text{ for the 1st integral, so that } du = -\sin x \, dx,$$

$$\text{and let } v = \sin x \text{ for the 2nd integral, so that } dv = \cos x \, dx$$

$$\text{Then } I = -3 \int u^3 du - \int v^3 dv$$

$$= -\frac{3}{4}\cos^4 x - \frac{1}{4}\sin^4 x$$

$$= -\frac{1}{4}(3\cos^4 x + \sin^4 x)$$

[As a check, we can substitute values of x such as 0 & $\frac{\pi}{2}$, to compare the answers by the two methods. There is in fact a difference of $\frac{3}{8}$ in each case, but this is due to the constant of integration.]

Method 3

By Parts, integrating $\cos x$ and differentiating $\sin 3x$ (for example):

$$I = \sin x \sin 3x - \int \sin x (3\cos 3x) dx$$

Then integrating $\sin x$ and differentiating $\cos 3x$,

$$I = \sin x \sin 3x - 3\{-\cos x \cos 3x - \int (-\cos x)(-3\sin 3x) dx\}$$

$$= \sin x \sin 3x + 3\cos x \cos 3x + 9I$$

$$\text{and hence } I = -\frac{1}{8}(\sin x \sin 3x + 3\cos x \cos 3x)$$

[Substituting values of x shows agreement with Method 1.]

$$(23) \int \operatorname{cosec} x \, dx$$

Solution

Method 1

$$I = \int \frac{1}{\sin x} \, dx = \int \frac{\sin x}{\sin^2 x} \, dx = \int \frac{\sin x}{1 - \cos^2 x} \, dx$$

Let $u = \cos x$, so that $du = -\sin x \, dx$

$$\text{Then } I = - \int \frac{1}{1-u^2} \, du = -\frac{1}{2} \int \frac{1}{1-u} + \frac{1}{1+u} \, du$$

$$= -\frac{1}{2} (\ln|1+u| - \ln|1-u|)$$

$$= -\frac{1}{2} \ln \left| \frac{1+\cos x}{1-\cos x} \right|$$

Method 2

$$\text{Let } t = \tan \left(\frac{x}{2} \right), \text{ so that } \tan x = \tan \left(\frac{x}{2} + \frac{x}{2} \right) = \frac{2t}{1-t^2}$$

Then it can be shown that $2t$, $1-t^2$ & $1+t^2$ form the sides of a right-angled triangle, so that $\operatorname{cosec} x = \frac{1+t^2}{2t}$

$$\text{Also, } dt = \sec^2 \left(\frac{x}{2} \right) \left(\frac{1}{2} \right) dx = \frac{1}{2} (1+t^2) dx$$

$$\text{so that } I = \int \frac{1+t^2}{2t} \left(\frac{2}{1+t^2} \right) dt = \int \frac{1}{t} dt = \ln \left| \tan \left(\frac{x}{2} \right) \right|$$

[To reconcile these two answers:

$$\text{rtp (result to prove): } \ln \left| \frac{1-\cos x}{1+\cos x} \right| = 2 \ln \left| \tan \left(\frac{x}{2} \right) \right|$$

$$\Leftrightarrow \frac{1-\cos x}{1+\cos x} = \tan^2 \left(\frac{x}{2} \right)$$

$$LHS = \frac{1 - \left(\frac{1-t^2}{1+t^2} \right)}{1 + \left(\frac{1-t^2}{1+t^2} \right)} = \frac{(1+t^2) - (1-t^2)}{(1+t^2) + (1-t^2)} = t^2 = RHS]$$

$$(24) \int \tan^3 x \, dx$$

Solution

$$I = \int (\sec^2 x - 1) \tan x \, dx = \int \sec^2 x \tan x \, dx - \int \tan x \, dx$$

For the 1st integral, $\int \sec^2 x \, dx = \tan x$, so let $u = \tan x$

Then $du = \sec^2 x \, dx$ and $I = \int u \, du - \ln|\sec x|$

$$= \frac{1}{2} \tan^2 x - \ln|\sec x|$$

$$(25) \int \tan^4 x \, dx$$

Solution

$$I = \int (\sec^2 x - 1) \tan^2 x \, dx$$

For the 1st integral, $\int \sec^2 x \, dx = \tan x$, so let $u = \tan x$

Then $du = \sec^2 x \, dx$ and $I = \int u^2 \, du - \int \sec^2 x - 1 \, dx$

$$= \frac{1}{3} \tan^3 x - \tan x + x$$

$$(26) \int \frac{\cos^3 x}{\sin^2 x} \, dx$$

Solution

$$I = \int \frac{\cos x (1 - \sin^2 x)}{\sin^2 x} \, dx$$

Let $u = \sin x$, so that $du = \cos x \, dx$,

$$\text{and } I = \int \frac{1-u^2}{u^2} \, du = \int u^{-2} - 1 \, du = -u^{-1} - u = -\operatorname{cosec} x - \sin x$$

$$(27) \int \frac{\cos x + \sin x}{\cos x - \sin x} dx$$

Solution

As $\int \cos x + \sin x dx = \sin x - \cos x$,

let $u = \sin x - \cos x$, so that $du = \cos x + \sin x dx$,

and $I = - \int \frac{1}{u} du = -\ln|\sin x - \cos x|$

$$(28) \int \tan x \sec x dx$$

Solution

$$I = \int \frac{\sin x}{\cos^2 x} dx$$

Let $u = \cos x$, so that $du = -\sin x dx$,

and $I = - \int u^{-2} du = u^{-1} = \sec x$

$$(29) \int \arcsin\left(\frac{x}{3}\right) dx$$

Solution

By Parts, integrating 1 and differentiating $\arcsin\left(\frac{x}{3}\right)$:

$$I = x \arcsin\left(\frac{x}{3}\right) - \int x \left(\frac{1}{\sqrt{1 - \left(\frac{x}{3}\right)^2}} \right) \left(\frac{1}{3}\right) dx$$

$$= x \arcsin\left(\frac{x}{3}\right) - \int \frac{x}{\sqrt{9 - x^2}} dx$$

Let $u = x^2$, so that $du = 2x dx$

$$\text{Then } I = x \arcsin\left(\frac{x}{3}\right) - \frac{1}{2} \int \frac{1}{\sqrt{9 - u}} du$$

$$\begin{aligned}
&= x \arcsin\left(\frac{x}{3}\right) - \frac{1}{2}(9 - u)^{\frac{1}{2}} \left(\frac{1}{\frac{1}{2}}\right) (-1) \\
&= x \arcsin\left(\frac{x}{3}\right) + (9 - x^2)^{\frac{1}{2}}
\end{aligned}$$

$$(30)^* \int \frac{\sin x}{\sin x + \cos x} dx$$

Solution

Note that $\int \frac{\sin x - \cos x}{\sin x + \cos x} dx = -\ln|\sin x + \cos x|$,

as $\frac{d}{dx}(\sin x + \cos x) = \cos x - \sin x$

Let $\sin x = A(\sin x + \cos x) + B(\sin x - \cos x)$

Then $A + B = 1$ & $A - B = 0$, so $A = B = \frac{1}{2}$

$$\begin{aligned}
\text{and } I &= \frac{1}{2} \int \frac{\sin x + \cos x}{\sin x + \cos x} + \frac{\sin x - \cos x}{\sin x + \cos x} dx \\
&= \frac{1}{2}x - \frac{1}{2} \ln|\sin x + \cos x|
\end{aligned}$$

(31) Find a reduction formula for $I_n = \int_0^1 x^n \sqrt{1 - x^2} dx$

Solution

By Parts, integrating $x\sqrt{1 - x^2}$ and differentiating x^{n-1} ,

$$\begin{aligned}
I_n &= \left[-\frac{1}{2} \frac{(1-x^2)^{\frac{3}{2}}}{\left(\frac{3}{2}\right)} x^{n-1} \right]_0^1 - \int_0^1 -\frac{1}{2} \frac{(1-x^2)^{\frac{3}{2}}}{\left(\frac{3}{2}\right)} (n-1)x^{n-2} dx \\
&= -\frac{1}{3} \left[(1-x^2)^{\frac{3}{2}} x^{n-1} \right]_0^1 + \frac{1}{3} (n-1) \int_0^1 (1-x^2)^{\frac{3}{2}} x^{n-2} dx \\
&= -\frac{1}{3} (0 - 0) + \frac{1}{3} (n-1) \int_0^1 (1-x^2) \sqrt{1-x^2} x^{n-2} dx
\end{aligned}$$

$$= \frac{1}{3}(n-1)\{I_{n-2} - I_n\}$$

$$\text{so that } I_n(3 + (n-1)) = (n-1)I_{n-2}$$

$$\text{and hence } I_n = \frac{n-1}{n+2}I_{n-2}$$

$$(32) \int_1^e (\ln x)^2 dx$$

Solution

By Parts, integrating 1 and differentiating $(\ln x)^2$:

$$\begin{aligned} I &= [x(\ln x)^2]_1^e - \int_1^e x(2\ln x) \left(\frac{1}{x}\right) dx \\ &= e - 2 \int_1^e \ln x dx \end{aligned}$$

Then, by Parts again, integrating 1 and differentiating $\ln x$:

$$\begin{aligned} I &= e - 2\{[x\ln x]_1^e - \int_1^e x \left(\frac{1}{x}\right) dx\} \\ &= e - 2e + 2(e-1) = e - 2 \end{aligned}$$

$$(33) \int 2^x dx$$

Solution

$$\text{Let } 2 = e^k, \text{ so that } I = \int e^{kx} dx = \frac{1}{k}e^{kx} = \frac{1}{\ln 2}2^x$$

$$(34) \int \frac{1}{x^2+6x+18} dx$$

Solution

$$I = \int \frac{1}{(x+3)^2+9} dx = \frac{1}{3} \arctan\left(\frac{x+3}{3}\right)$$

$$(35) \int \frac{x^2}{1+x^6} dx$$

Solution

Let $u = x^3$, so that $du = 3x^2 dx$,

$$\text{and } I = \frac{1}{3} \int \frac{1}{1+u^2} du = \frac{1}{3} \arctan u = \frac{1}{3} \arctan(x^3)$$

$$(36) \int \frac{1}{(2x^2+3)^{\frac{3}{2}}} dx$$

Solution

Let $2x^2 = 3\tan^2\theta$, so that $x = \sqrt{\frac{3}{2}} \tan\theta$ and $dx = \sqrt{\frac{3}{2}} \sec^2\theta d\theta$

$$\text{Then } I = \sqrt{\frac{3}{2}} \int \frac{\sec^2\theta}{3^{\frac{3}{2}}(\tan^2\theta+1)^{\frac{3}{2}}} d\theta$$

$$= \frac{1}{3\sqrt{2}} \int \frac{\sec^2\theta}{\sec^3\theta} d\theta = \frac{1}{3\sqrt{2}} \int \cos\theta d\theta = \frac{1}{3\sqrt{2}} \sin\theta$$

$$\text{Now } \sin^2\theta = 1 - \cos^2\theta = 1 - \frac{1}{1+\tan^2\theta} = 1 - \frac{1}{1+\frac{2}{3}x^2}$$

$$= 1 - \frac{3}{3+2x^2} = \frac{2x^2}{3+2x^2}$$

$$\text{so that } I = \frac{1}{3\sqrt{2}} \sqrt{\frac{2x^2}{3+2x^2}} = \frac{x}{3\sqrt{3+2x^2}}$$

$$(37)^{**} \int \sqrt{4x^2 - 1} \, dx$$

Solution

Let $2x = \sec\theta$, so that $2dx = \tan\theta \sec\theta \, d\theta$

$$\text{Then } I = \int \sqrt{\sec^2\theta - 1} \left(\frac{1}{2}\right) \tan\theta \sec\theta \, d\theta$$

$$= \frac{1}{2} \int \tan^2\theta \sec\theta \, d\theta$$

$$= \frac{1}{2} \int (\sec^2\theta - 1) \sec\theta \, d\theta$$

$$= \frac{1}{2} \int \sec^3\theta \, d\theta - \frac{1}{2} \int \sec\theta \, d\theta$$

Then see (4) for $\int \sec\theta \, d\theta$ and (17) for $\int \sec^3\theta \, d\theta$.

Alternative method: See Integration Exercises - Part 3 (Hyperbolic Functions).

$$(38) \int \frac{4x+5}{\sqrt{4-6x-x^2}} \, dx$$

Solution

$$I = -2 \int \frac{-6-2x}{\sqrt{4-6x-x^2}} \, dx - \int \frac{7}{\sqrt{4-6x-x^2}} \, dx$$

$$= -\frac{2(4-6x-x^2)^{\frac{1}{2}}}{\left(\frac{1}{2}\right)} - 7 \int \frac{1}{\sqrt{13-(x+3)^2}} \, dx$$

$$= -4(4-6x-x^2)^{\frac{1}{2}} - 7 \arcsin\left(\frac{x+3}{\sqrt{13}}\right)$$

$$(39) \int \frac{\sin \sqrt{x}}{\sqrt{x}} dx$$

Solution

Let $y = \sqrt{x}$, so that $dy = \frac{1}{2} x^{-\frac{1}{2}} dx$

and $I = \int \sin y \cdot 2 dy = -2 \cos \sqrt{x}$

(40) Find a reduction formula for $I_n = \int_0^2 x^n \sqrt{4-x^2} dx$, and hence show that $\int_0^2 x^5 \sqrt{4-x^2} dx = \frac{1024}{105}$

Solution

By Parts, integrating $x\sqrt{4-x^2}$ and differentiating x^{n-1} ,

$$\begin{aligned} I_n &= \left[-\frac{1}{2} \frac{(4-x^2)^{\frac{3}{2}}}{\left(\frac{3}{2}\right)} x^{n-1} \right]_0^2 - \int_0^2 -\frac{1}{2} \frac{(4-x^2)^{\frac{3}{2}}}{\left(\frac{3}{2}\right)} (n-1) x^{n-2} dx \\ &= -\frac{1}{3} \left[(4-x^2)^{\frac{3}{2}} x^{n-1} \right]_0^2 + \frac{1}{3} (n-1) \int_0^2 (4-x^2)^{\frac{3}{2}} x^{n-2} dx \\ &= -\frac{1}{3} (0-0) + \frac{1}{3} (n-1) \int_0^2 (4-x^2) \sqrt{4-x^2} x^{n-2} dx \\ &= \frac{1}{3} (n-1) \{4I_{n-2} - I_n\} \end{aligned}$$

so that $I_n(3 + (n-1)) = 4(n-1)I_{n-2}$

and hence $I_n = \frac{4(n-1)}{n+2} I_{n-2}$

Then $\int_0^2 x^5 \sqrt{4-x^2} dx = I_5 = \frac{4(4)}{7} I_3 = \frac{16}{7} \frac{4(2)}{5} I_1 = \frac{128}{35} I_1$

and $I_1 = \int_0^2 x \sqrt{4-x^2} dx = -\frac{1}{2} \int_0^2 -2x \sqrt{4-x^2} dx$

$$= -\frac{1}{2} \left[\frac{(4-x^2)^{\frac{3}{2}}}{\left(\frac{3}{2}\right)} \right]_0^2 = -\frac{1}{3} (0-8) = \frac{8}{3},$$

so that $\int_0^2 x^5 \sqrt{4-x^2} \, dx = \frac{128}{35} \left(\frac{8}{3}\right) = \frac{1024}{105}$

(41) Find a reduction formula for $I_n = \int_0^\pi \cos^n x \, dx$, and hence show that $\int_0^\pi \cos^4 x \, dx = \frac{3\pi}{8}$

Solution

$$I_n = \int_0^\pi \cos x \cdot \cos^{n-1} x \, dx$$

Integrating by Parts (integrating $\cos x$ and differentiating $\cos^{n-1} x$),

$$I_n = [\sin x \cdot \cos^{n-1} x]_0^\pi - \int_0^\pi \sin x (n-1) \cos^{n-2} (-\sin x) dx$$

$$= 0 + (n-1) \int_0^\pi (1 - \cos^2 x) \cos^{n-2} dx$$

$$= (n-1)I_{n-2} - (n-1)I_n,$$

so that $nI_n = (n-1)I_{n-2}$

and $I_n = \frac{n-1}{n} I_{n-2}$

Hence $\int_0^\pi \cos^4 x = \frac{3}{4} I_2 = \frac{3}{4} \cdot \frac{1}{2} I_0 = \frac{3}{8} \int_0^\pi 1 \, dx = \frac{3}{8} [x]_0^\pi = \frac{3}{8} (\pi - 0) = \frac{3\pi}{8}$

(42) Use the substitution $t = \tan\left(\frac{x}{2}\right)$ to find $\int_0^{\frac{\pi}{2}} \frac{1}{2+\sin x} \, dx$

Solution

Let $t = \tan\left(\frac{x}{2}\right)$, so that $\tan x = \frac{2t}{1-t^2}$

[and the right-angled triangle with angle x has sides

$1 - t^2, 2t$ & $1 + t^2$]

and $dt = \frac{1}{2} \sec^2\left(\frac{x}{2}\right) dx = \frac{1}{2}(t^2 + 1)dx$

$$\begin{aligned}
\text{Then } I &= \int_0^{\frac{\pi}{2}} \frac{1}{2+\sin x} dx = \int_0^1 \frac{1}{2+\left(\frac{2t}{1+t^2}\right)} \cdot \frac{2}{t^2+1} dt \\
&= \int_0^1 \frac{1}{(t^2+1)+t} dt \\
&= \int_0^1 \frac{1}{\left(t+\frac{1}{2}\right)^2+\frac{3}{4}} dt \\
&= \left[\frac{1}{\left(\frac{\sqrt{3}}{2}\right)} \arctan\left(\frac{t+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) \right]_0^1 \\
&= \frac{2}{\sqrt{3}} (\arctan(\sqrt{3}) - \arctan\left(\frac{1}{\sqrt{3}}\right)) \\
&= \frac{2}{\sqrt{3}} \left(\frac{\pi}{3} - \frac{\pi}{6}\right) \\
&= \frac{2\pi}{6\sqrt{3}} = \frac{\pi\sqrt{3}}{9}
\end{aligned}$$

$$(43) \int \operatorname{cosec} x \sec x dx$$

Solution

$$(1) I = \int \frac{1}{\sin x \cos x} dx = \int \frac{1}{\left(\frac{1}{2}\right) \sin 2x} dx = 2 \int \operatorname{cosec} 2x dx$$

Writing $u = 2x$, so that $du = 2dx$,

$$\begin{aligned}
I &= 2 \int \operatorname{cosec} u \cdot \left(\frac{1}{2}\right) du \\
&= \ln \left| \tan \left(\frac{u}{2}\right) \right|, \text{ from tables} \\
&= \ln |\tan x|
\end{aligned}$$

$$(2) I = \int \frac{1}{\sin x \cos x} dx = \int \frac{\cos x}{\sin x (1 - \sin^2 x)} dx$$

Let $u = \sin x$, so that $du = \cos x dx$

$$\text{and } I = \int \frac{1}{u(1-u^2)} du \quad (*)$$

$$\text{Then let } z = \frac{1}{u}, \text{ so that } dz = -\frac{1}{u^2} du$$

$$\text{and } I = \int \frac{z}{1-\frac{1}{z^2}} \left(-\frac{1}{z^2}\right) dz = -\int \frac{z}{z^2-1} dz$$

$$= -\frac{1}{2} \ln|z^2 - 1|$$

$$= -\frac{1}{2} \ln\left|\frac{1}{u^2} - 1\right|$$

$$= -\frac{1}{2} \ln|\operatorname{cosec}^2 x - 1|$$

$$= -\frac{1}{2} \ln|\cot^2 x|$$

$$= \ln|\tan x|$$

$$(3) \text{ From } (*) \text{ in } (2), \int \frac{1}{u(1-u^2)} du = \int \frac{A}{u} + \frac{B}{1-u} + \frac{C}{1+u} du$$

$$\text{where } A(1-u)(1+u) + Bu(1+u) + Cu(1-u) = 1$$

$$\text{Then } u = 0 \Rightarrow A = 1; u = 1 \Rightarrow 2B = 1; u = -1 \Rightarrow -2C = 1,$$

$$\text{so that } I = \int \frac{1}{u} + \frac{\left(\frac{1}{2}\right)}{1-u} - \frac{\left(\frac{1}{2}\right)}{1+u} du$$

$$= \ln|u| - \frac{1}{2} \ln|1-u| - \frac{1}{2} \ln|1+u|$$

$$= \frac{1}{2} \ln\left|\frac{u^2}{(1-u)(1+u)}\right| \quad (\text{strictly speaking, this use of the modulus}$$

would need to be justified)

$$= \frac{1}{2} \ln\left|\frac{\sin^2 x}{(1-\sin x)(1+\sin x)}\right|$$

$$= \frac{1}{2} \ln\left|\frac{\sin^2 x}{\cos^2 x}\right|$$

$$= \ln|\tan x|]$$

(4) Let $t = \tan\left(\frac{x}{2}\right)$, so that $\tan x = \frac{2t}{1-t^2}$

[and the right-angled triangle with angle x has sides

$$1 - t^2, 2t \text{ \& } 1 + t^2]$$

and $dt = \frac{1}{2} \sec^2\left(\frac{x}{2}\right) dx = \frac{1}{2}(t^2 + 1)dx$

Then $I = \int \operatorname{cosec} x \sec x \, dx = \int \frac{1+t^2}{2t} \cdot \frac{1+t^2}{1-t^2} \cdot \frac{2}{t^2+1} \, dt$

$$= \int \frac{1+t^2}{t(1-t^2)} \, dt$$

$$= \int \frac{1-t^2}{t(1-t^2)} \, dt + \int \frac{2t^2}{t(1-t^2)} \, dt$$

$$= \int \frac{1}{t} \, dt + 2 \int \frac{t}{(1-t^2)} \, dt$$

$$= \ln|t| - \ln|1 - t^2|$$

$$= \ln \left| \frac{t}{1-t^2} \right|$$

$$= \ln \left| \frac{1}{2} \tan x \right|$$

$$= \ln|\tan x| + \ln\left(\frac{1}{2}\right)$$

ie $\ln|\tan x|$, apart from a constant

(44) $\int \operatorname{cosec}^2 x \, dx$

Solution

[As $\int \sec^2 x \, dx = \tan x$, we might suspect that the answer was

either $\cot x$ or $-\cot x$, and we could confirm this by

differentiation.]

$$\begin{aligned}
 \int \operatorname{cosec}^2 x \, dx &= \int \frac{1}{\sin^2 x} \, dx \\
 &= \int \frac{1}{\cos^2(\frac{\pi}{2}-x)} \, dx \\
 &= \int \sec^2\left(\frac{\pi}{2}-x\right) \, dx \\
 &= -\tan\left(\frac{\pi}{2}-x\right) \\
 &= -\cot x
 \end{aligned}$$

$$(45) J = \int_0^{\frac{\pi}{2}} \sin^4 x \, dx$$

Solution

$$\begin{aligned}
 (1) \text{ Consider } \int_0^{\frac{\pi}{2}} (\sin^2 x + \cos^2 x)^2 \, dx &= J + \int_0^{\frac{\pi}{2}} (\cos x)^4 \, dx \\
 &+ 2 \int_0^{\frac{\pi}{2}} \sin^2 x \cos^2 x \, dx \quad (*)
 \end{aligned}$$

Considering the areas under the curves of $\sin^4 x$ and $\cos^4 x$ (or making the substitution $y = \frac{\pi}{2} - x$ in $\int_0^{\frac{\pi}{2}} (\sin x)^4 \, dx$),

$$\int_0^{\frac{\pi}{2}} (\cos x)^4 \, dx = J,$$

$$\text{and } (*) \text{ becomes } \int_0^{\frac{\pi}{2}} 1 \, dx = 2J + \frac{1}{2} \int_0^{\frac{\pi}{2}} (\sin 2x)^2 \, dx$$

$$\text{so that } 2J = \frac{\pi}{2} - \frac{1}{4} \int_0^{\frac{\pi}{2}} (1 - \cos 4x) \, dx,$$

$$\text{and } J = \frac{\pi}{4} - \frac{1}{8} \left[x - \frac{1}{4} \sin 4x \right]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{4} - \frac{1}{8} \left(\frac{\pi}{2} \right) = \frac{3\pi}{16},$$

$$\begin{aligned}
(2) J &= \int_0^{\frac{\pi}{2}} \left(\frac{1}{2}[1 - \cos 2x]\right)^2 dx \\
&= \frac{1}{4} \int_0^{\frac{\pi}{2}} 1 - 2\cos 2x + (\cos 2x)^2 dx \\
&= \frac{1}{4} [x - \sin 2x]_0^{\frac{\pi}{2}} + \frac{1}{4} \int_0^{\frac{\pi}{2}} \frac{1}{2} [1 + \cos 4x] dx \\
&= \frac{1}{4} \left(\frac{\pi}{2}\right) + \frac{1}{8} [x + \frac{1}{4} \sin 4x]_0^{\frac{\pi}{2}} \\
&= \frac{\pi}{8} + \frac{\pi}{16} \\
&= \frac{3\pi}{16}
\end{aligned}$$