

Integration Theory (7 pages; 13/1/25)

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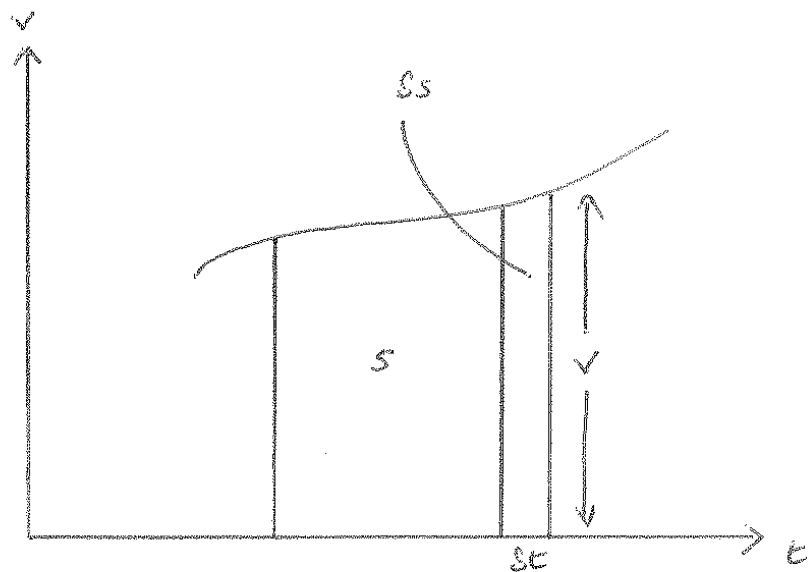
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(A) The two interpretations of integration

Integration can be interpreted as either the area under a curve, or as the opposite of differentiation. To show how these two interpretations can be reconciled, refer to the diagram below.



v & s can be interpreted as speed and displacement, but the argument holds for other situations. s is defined to be the area under the curve of v , and, by the first definition of integration,

$$s = \int v \, dt \quad (\text{A})$$

(to work out a specific area, limits would obviously be needed).

We want to show that integration is also the opposite of differentiation. This will be the case if $\frac{ds}{dt} = v$

From the diagram, $\frac{ds}{dt}$ is the rate at which the area increases, and is the limit as $\delta t \rightarrow 0$ of $\frac{\delta s}{\delta t}$, which equals v , since $\delta s \rightarrow v\delta t$ as $\delta t \rightarrow 0$. Thus we have shown that $\frac{ds}{dt} = v$.

In the case where v & s are speed and displacement, this works because speed is the rate of change of displacement, and displacement = speed $\times$ time if the speed is constant (so that the displacement is the area under a horizontal line), and the natural extension of this is for the displacement to be the area under the speed-time graph in the case of a varying speed.

(B) Indefinite integration

In the definite integral $\int_{t_1}^{t_2} v(t)dt$, t is appearing as a parameter (which ranges from t_1 to t_2). It can just as easily be written as

$$\int_{t_1}^{t_2} v(x)dx$$

If t_2 is now considered to be a variable value of t , so that the definite integral represents the area under the curve as a function of t_2 , then, writing t instead of t_2 : $\int_{t_1}^t v(x)dx = s(t) - s(t_1)$

(where $v(x)$ is the derivative of $s(x)$; eg speed and displacement, respectively).

The integral is now a function of t (whereas the definite integral

$\int_{t_1}^{t_2} v(t)dt$ was a fixed value).

It is termed an 'indefinite' integral and, by convention, the following notation is adopted: $\int v(t)dt = s(t) + C$

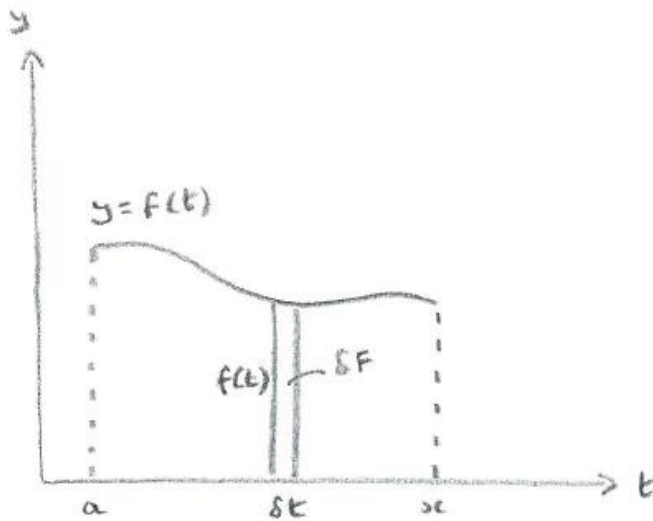
C in effect equals $-s(t_1)$ and is a constant ; ie not changing with t (C is the 'constant of integration'). It can take any value (including positive values, since $s(t_1)$ can generally be made to be negative).

Note that t has been reintroduced on the left hand side, as it can no longer be confused with the upper limit of integration. This notation is slightly unsatisfactory, since the t on the left hand side is a parameter over which the integration is being carried out, whereas the t on the right hand side is the upper limit of the integration. However, the t on the left hand side does serve to indicate that the integral is to be a function of t .

(C) Fundamental Theorem of Calculus

If $F(x) = \int_a^x f(t)dt$, then $F'(x) = f(x)$.

It amounts to saying that integration is the opposite of differentiation.



$$\delta F \approx f(t)\delta t \Rightarrow \frac{\delta F}{\delta t} \approx f(t)$$

$$F'(t) = \frac{dF}{dt} = \lim_{\delta t \rightarrow 0} \frac{\delta F}{\delta t} = f(t)$$

and at $t = x$, $F'(x) = f(x)$

$$(D) \int \frac{1}{x} dx = \ln |x|$$

Given that $\int \frac{1}{x} dx = \ln x$ for $x > 0$, it can be shown that

$$\int \frac{1}{x} dx = \ln|x| \text{ for all } x \neq 0$$

Method 1

If $\int \frac{1}{x} dx = \ln x$ for $x > 0$, then $\frac{d}{dx}(\ln x) = \frac{1}{x}$ for $x > 0$

For the case where $x < 0$:

Let $y = -x$, so that $\frac{d}{dy}(\ln y) = \frac{1}{y}$, as $y > 0$

[To convert back to x s:]

Then, as $\frac{d}{dy}(\ln y) = \frac{d}{dx}(\ln y) \cdot \frac{dx}{dy}$,

it follows that $\frac{d}{dx}(\ln y) \cdot \frac{dx}{dy} = \frac{1}{(-x)}$

giving $\frac{d}{dx}(\ln[-x])(-1) = \frac{1}{(-x)}$

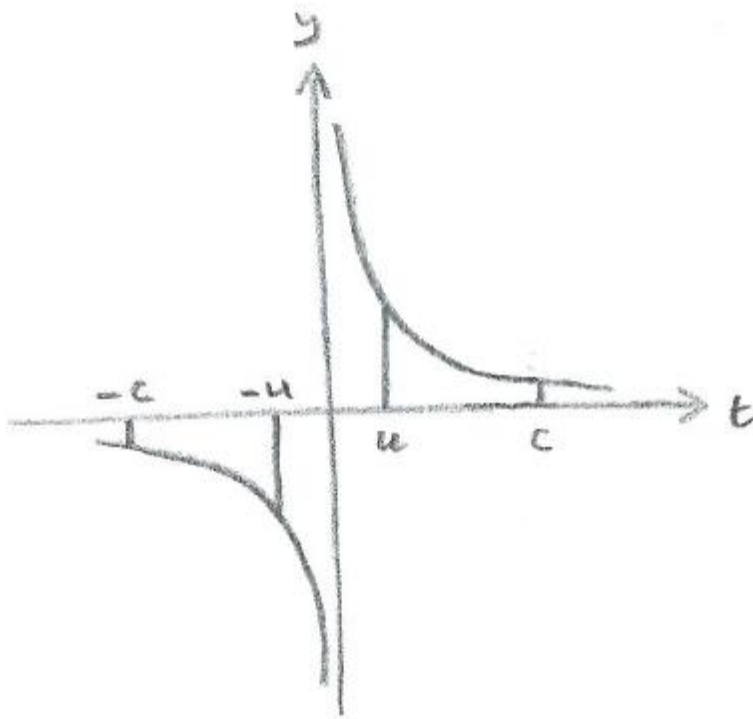
and so $\frac{d}{dx}(\ln|x|) = \frac{1}{x}$ for $x < 0$ (*)

and therefore $\int \frac{1}{x} dx = \ln|x|$ for $x < 0$, as well as $x > 0$

[Note that the function $y = \ln|x|$ for $x < 0$ is the reflection in the y -axis of $y = \ln x$ (for $x > 0$), and therefore has a negative gradient, which agrees with (*).]

Method 2

Referring to the diagram below, where $u = -x > 0$ & $c > 0$,



$$\int_{-c}^x \frac{1}{t} dt = \int_{-c}^{-u} \frac{1}{t} dt$$

= - (positive) area between graph and t -axis on LHS

= - (positive) area between graph and t -axis on RHS

$$= - \int_u^c \frac{1}{t} dt = \int_c^u \frac{1}{t} dt = \ln u - \ln c$$

As $\int \frac{1}{x} dx$ only differs from $\int_{-c}^x \frac{1}{t} dt$ by an arbitrary constant, it follows that, when $x < 0$, $\int \frac{1}{x} dx = \ln u + C = \ln|-x| + C$, as required.

(E) Substitutions

Example: $I = \int \frac{\sin x}{\cos x} dx$

Informal approach (but acceptable for exam purposes)

Let $u = \cos x$, so that $du = -\sin x dx$,

and then $I = - \int \frac{1}{u} du = -\ln u + C = -\ln(\cos x) + C$

$$= \ln(\sec x) + C$$

More rigorous approach

Let $u = \cos x$

[Result to prove: $I = \int \frac{\sin x}{\cos x} \frac{dx}{du} du$;

then, as $\frac{dx}{du} = \frac{1}{(\frac{du}{dx})} = \frac{1}{-\sin x}$, $I = - \int \frac{1}{u} du]$

Now, $I = \int \frac{\sin x}{\cos x} dx \Rightarrow \frac{dI}{dx} = \frac{\sin x}{\cos x}$

and $\frac{dI}{du} = \frac{dI}{dx} \cdot \frac{dx}{du} = \frac{\sin x}{\cos x} \cdot \frac{dx}{du}$,

so that $I = \int \frac{\sin x}{\cos x} \frac{dx}{du} du$

Then $\frac{dx}{du} = \frac{1}{\left(\frac{du}{dx}\right)} = \frac{1}{-\sin x}$,

so that $I = - \int \frac{1}{u} du$ etc

(F) Mean value of the function $f(x)$ is $\frac{1}{b-a} \int_a^b f(x) dx$