Integration – Miscellaneous: Exercises (with Sol'ns)

(7 pages; 27/3/26)

(1) Find the values of the following integrals, or show that they are not defined.

(i) $\int_{-2}^{\infty} \frac{1}{x^2} \, dx$

(ii) $\int_{-\infty}^{-\frac{1}{2}} e^{2x} \, dx$

(iii) $\int_{0}^{1} x^{-\frac{2}{3}} \, dx$

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Solutions

(i) $\int_{-2}^{\infty} \frac{1}{x^2} dx = \lim_{a \to 0^-} \int_{-2}^{a} \frac{1}{x^2}\, dx + \lim_{\substack{b \to 0^+ \\ c \to \infty}} \int_b^c \frac{1}{x^2}\, dx$

$= \lim_{a \to 0^-} [-\frac{1}{x}]_{-2}^{a} + \lim_{\substack{b \to 0^+ \\ c \to \infty}} [-\frac{1}{x}]_b^c$

which is undefined, as $\frac{1}{a}$ is undefined as $a \to 0^-$

(and similarly for b)

(ii) $\int_{-\infty}^{-\frac{1}{2}} e^{2x}\, dx = \lim_{a \to -\infty} \int_a^{-\frac{1}{2}} e^{2x}\, dx$

$= \lim_{a \to -\infty} [\frac{1}{2} e^{2x}]_a^{-\frac{1}{2}}$]

$= \lim_{a \to -\infty} \frac{1}{2}(e^{-1} - e^{2a})$

$= \frac{1}{2e}$

(iii) $\int_0^1 x^{-\frac{2}{3}}\, dx = \lim_{a \to 0^+} \int_a^1 x^{-\frac{2}{3}}\, dx$

$= \lim_{a \to 0^+} \left[3x^{\frac{1}{3}}\right]_a^1 = \lim_{a \to 0^+} \left(3 - 3a^{\frac{1}{3}}\right)$

$= 3$

(2) Explain the following 'paradox':

$\int \frac{1}{2\mathrm{x}} dx = \frac{1}{2} \int \frac{1}{x} dx = \frac{1}{2} lnx + C$

but $\int \frac{1}{2\mathrm{x}} dx = \frac{1}{2} \ln(2x) + C$ (by the reverse Chain rule)

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Solution

ln $(2x)$ can be written as $ln2 + lnx$, giving the first form of the answer, after renaming the constant

(3) Given that $\int \frac{1}{x} dx = \ln x$ for $x > 0$, show that $\int \frac{1}{x} dx = ln|x|$ for all $x \neq 0$

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Solution

Method 1

If $\int \frac{1}{x} dx = \ln x$ for $x > 0$, then $\frac{d}{dx}(lnx) = \frac{1}{x}$ for $x > 0$

For the case where $x < 0$:

Let $y = -x$, so that $\frac{d}{dy}(lny) = \frac{1}{y}$, as $y > 0$

[To convert back to xs:]

Then, as $\frac{d}{dy}(lny) = \frac{d}{dx}(lny).\frac{dx}{dy}$,

it follows that $\frac{d}{dx}(lny).\frac{dx}{dy} = \frac{1}{(-x)}$

giving $\frac{d}{dx}(ln[-x])(-1) = \frac{1}{(-x)}$

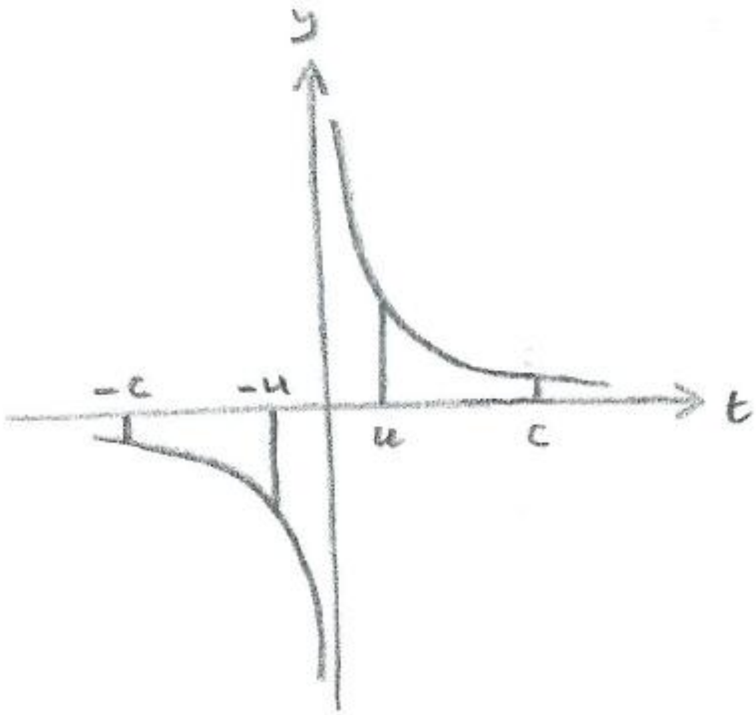
and so $\frac{d}{dx}(\ln|x|) = \frac{1}{x}$ for $x < 0$ (*)

and therefore $\int \frac{1}{x} dx = \ln|x|$ for $x < 0$, as well as $x > 0$

[Note that the function $y = \ln$ |x| for $x < 0$ is the reflection in the y-axis of $y = \ln x$ $(for\ x > 0)$, and therefore has a negative gradient, which agrees with (*).]

Method 2

Referring to the diagram below, where $u = -x > 0$ & $c > 0$,



$\int_{-c}^{x} \frac{1}{t} dt = \int_{-c}^{-u} \frac{1}{t} dt$

$= -$ (positive) area between graph and t-axis on LHS

$= -$ (positive) area between graph and t-axis on RHS

$= -\int_{u}^{c} \frac{1}{t} dt = \int_{c}^{u} \frac{1}{t} dt = lnu - lnc$

As $\int \frac{1}{x} dx$ only differs from $\int_{-c}^{x} \frac{1}{t} dt$ by an arbitrary constant, it follows that, when $x < 0$, $\int \frac{1}{x} dx = ln\, u + C = \ln|-x| + C$

$\ln|x| + C$, as required.