Integration Exercises - Part 4 (Misc) (Sol'ns)

(6 pages; 8/8/19)

(1) If $\int_{-a}^{a} f(x)\, dx = b$, find $\int_{-a}^{a} f(-x)\, dx$

Solution

b also, as $f(-x)$ is the reflection of $f(x)$ in the y-axis

(2) Explain the following 'paradox':

$\int \frac{1}{2\text{x}} dx = \frac{1}{2}\int \frac{1}{x} dx = \frac{1}{2} lnx + C$

but $\int \frac{1}{2\text{x}} dx = \frac{1}{2}\ln(2x) + C$ (by the reverse Chain rule)

Solution

$\ln(2x)$ can be written as $ln2 + lnx$, giving the first form of the answer, after renaming the constant

(3) The region between the line $y = 6 - 2x$ and the curve $y = \frac{4}{x}$ is rotated about the y-axis through 360°. Find the exact volume generated.

Solution

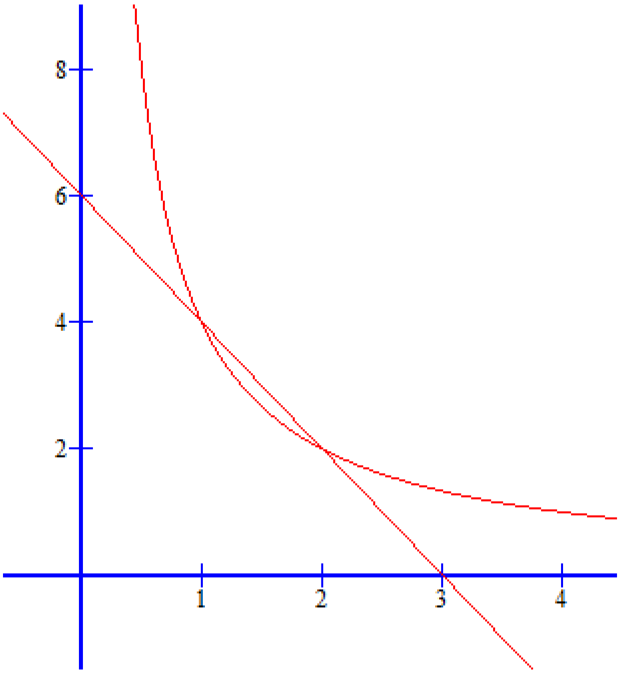
To find the points of intersection of the line and curve:

$y = 6 - 2x$ and $y = \frac{4}{x} \Rightarrow \frac{4}{x} = 6 - 2x$,

so that $4 = 6x - 2x^2$, or $x^2 - 3x + 2 = 0$

$\Rightarrow (x - 1)(x - 2) = 0$,

so that the points of intersection are (1,4) and (2,2).



For the line, $x = 3 - \frac{y}{2}$, and for the curve, $x = \frac{4}{y}$

The required volume is $\int_2^4 \pi \left\{\left(3 - \frac{y}{2}\right)^2 - \left(\frac{4}{y}\right)^2\right\} dy$

$= \pi \int_2^4 9 - 3y + \frac{y^2}{4} - \frac{16}{y^2}\ dy$

$= \pi[9y - \frac{3y^2}{2} + \frac{y^3}{12} + \frac{16}{y}]_2^4$

$= \pi\{\left(36 - 24 + \frac{16}{3} + 4\right) - \left(18 - 6 + \frac{2}{3} + 8\right)\}$

$= \pi\left\{-4 + \frac{14}{3}\right\} = \frac{2\pi}{3}\ units^3$

(4) The region between the parabola $y^2 = 4x$, the x-axis and the line $x = 1$ is rotated about the x-axis through 360°.

(i) Find the exact volume generated:

(a) by integrating with respect to x

(b) by integrating with respect to the parameter t, where $x = t^2$ and $y = 2t$

(ii) Use the mean value of the function to carry out a rough check on your answer in (i).

(iii) Find the curved surface area associated with the volume generated in (i):

(a) by integrating with respect to x

(b) by integrating with respect to y

(c) by integrating with respect to t

Solution

(i)(a) Volume $= \pi \int_0^1 y^2 dx = \pi \int_0^1 4x\, dx = \pi[2x^2]_0^1 =$
$2\pi(1-0) = 2\pi$

(b) $x = 0 \Rightarrow t = 0; x = 1 \Rightarrow t = 1$

Volume $= \pi \int_0^1 y^2 \frac{dx}{dt} dt = \pi \int_0^1 (2t)^2(2t) dt = 8\pi \int_0^1 t^3 dt$

$= 8\pi \left[\frac{1}{4}t^4\right]_0^1 = 2\pi(1-0) = 2\pi$

(ii) Mean value $= \frac{1}{1-0}\int_0^1 \sqrt{4x}\, dx = 2\int_0^1 x^{\frac{1}{2}} dx$

$= 2\left[\frac{x^{\frac{3}{2}}}{\left(\frac{3}{2}\right)}\right]_0^1 = \frac{4}{3}$

Approximate volume is that of a cylinder of radius $\frac{4}{3}$ and length 1;

ie $\pi\left(\frac{4}{3}\right)^2(1)=\frac{16}{9}\pi$, which is reasonably close to 2π.

(iii)(a) Curved SA $=\int_0^1 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

$y=\sqrt{4x}$ and $\frac{dy}{dx}=2(\frac{1}{2})x^{-\frac{1}{2}}$

so that $SA=4\pi\int_0^1 x^{\frac{1}{2}}\sqrt{1+\frac{1}{x}}\,dx=4\pi\int_0^1\sqrt{x+1}\,dx$

$=4\pi\left[\frac{(x+1)^{\frac{3}{2}}}{\left(\frac{3}{2}\right)}\right]_0^1=\frac{8\pi}{3}(2\sqrt{2}-1)$

(b) Curved SA $=\int_0^1 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

$=\int_{x=0}^{x=1}2\pi y\sqrt{dx^2+dy^2}$ (informally)

$=\int_0^2 2\pi y\sqrt{\left(\frac{dx}{dy}\right)^2+1}\,dy$

$y^2=4x$, so that $\frac{dx}{dy}=\frac{1}{4}(2y)$

and SA $=2\pi\int_0^2 y\sqrt{\frac{y^2}{4}+1}\,dy$

Then, as $\frac{d}{dy}\left(\frac{y^2}{4}\right)=\frac{2y}{4}=\frac{y}{2}$, SA $=4\pi\left[\frac{\left(\frac{y^2}{4}+1\right)^{\frac{3}{2}}}{\left(\frac{3}{2}\right)}\right]_0^2$

$=\frac{8\pi}{3}(2\sqrt{2}-1)$

(c) Curved SA $=\int_0^1 2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$

$x = t^2$ and $y = 2t$, so that $\frac{dx}{dt} = 2t$ and $\frac{dy}{dt} = 2$

and SA $= \int_0^1 2\pi(2t)\sqrt{4t^2 + 4}\,dt$

$= 4\pi\int_0^1 2t\sqrt{t^2 + 1}\,dt$

$= 4\pi\left[\frac{(t^2+1)^{\frac{3}{2}}}{\left(\frac{3}{2}\right)}\right]_0^1$

$= \frac{8\pi}{3}(2\sqrt{2} - 1)$

(5) The curve C has equation $y = \frac{1}{3}x^3 + \frac{1}{4x}$. The points A and B on C have x coordinates 1 and 2, respectively. Find the length of the arc from A to B.

Solution

Length of arc $= \int_1^2 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$

and $\frac{dy}{dx} = x^2 - \frac{1}{4x^2}$

Note that $2x^2\left(-\frac{1}{4x^2}\right) = -\frac{1}{2}$

so that $1 + 2x^2\left(-\frac{1}{4x^2}\right) = \frac{1}{2} = 2x^2\left(\frac{1}{4x^2}\right)$

and $1 + \left(x^2 - \frac{1}{4x^2}\right)^2 = \left(x^2 + \frac{1}{4x^2}\right)^2$

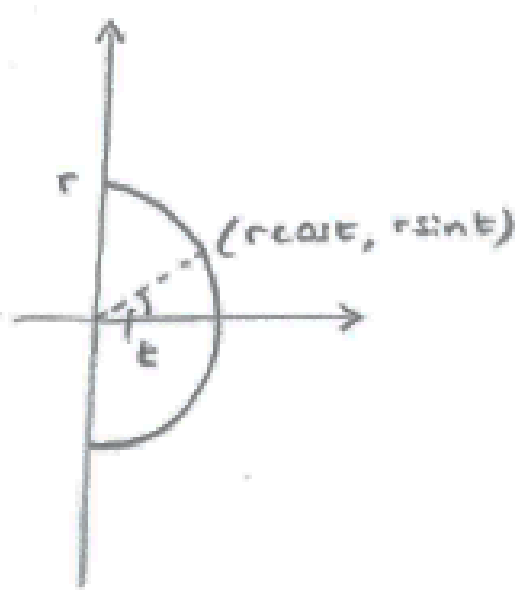
So length of arc $= \int_1^2 x^2 + \frac{1}{4x^2}\,dx$

$= \left[\frac{1}{3}x^3 - \frac{1}{4x}\right]_1^2 = \left(\frac{8}{3} - \frac{1}{8}\right) - \left(\frac{1}{3} - \frac{1}{4}\right) = \frac{59}{24}$

(6) Use integration with respect to a suitable parameter to show that the surface area of a sphere of radius r is $4\pi r^2$.

Solution

Consider the hemisphere obtained by rotating a quarter circle about the x-axis, as in the diagram below.



Surface area of hemisphere $= \int_{t_1}^{t_2} 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

$= \int_0^{\frac{\pi}{2}} 2\pi(rsint)\sqrt{(-rsint)^2 + (rcost)^2}\ \mathrm{dt}$

$= 2\pi r^2 \int_0^{\frac{\pi}{2}} sint\, dt$

$= 2\pi r^2[-cost]_0^{\frac{\pi}{2}}$

$= 2\pi r^2(0 - (-1))$

$= 2\pi r^2$

Then the surface area of the sphere is double this, to give $4\pi r^2$.