Inequalities - Exercises (Sol'ns)(6 pages; 9/1/20)

(1*) How would you solve the inequality: $\frac{1}{x} < x$?

Solution

Method 1: Multiply both sides by x^2

Method 2: Treat the cases $x < 0$ and $x > 0$ separately

Method 3: Rearrange as $\frac{1}{x} - x < 0$

Method 4: Sketch $y = \frac{1}{x}$ and $y = x$, and consider points of intersection

(2*) Is $\frac{6}{7} < \frac{2}{\sqrt{5}}$?

Solution

$\frac{6}{7} < \frac{2}{\sqrt{5}} \Leftrightarrow \frac{36}{49} < \frac{4}{5}$

$49 \times 0.8 = \frac{1}{10}(320 + 72) = 39.2 > 36$

So $\frac{36}{49} < \frac{39.2}{49} = 0.8 = \frac{4}{5}$

Answer is Yes.

(3*) Which is larger: $\frac{\sqrt{7}}{2}$ or $\frac{1+\sqrt{6}}{3}$ (without using a calculator)?

Solution

Considering the difference of squares:

$\frac{7}{4} - \frac{(1+2\sqrt{6}+6)}{9} = \frac{63-28-8\sqrt{6}}{36} > \frac{35-8(3)}{36} > 0$; so $\frac{\sqrt{7}}{2}$ is larger

[Another approach is to investigate $\frac{\left(\frac{7}{4}\right)}{\left(\frac{7+2\sqrt{6}}{9}\right)} = \frac{63(7-2\sqrt{6})}{4(49-24)} =$ $\frac{63(7-2\sqrt{6})}{100}$, but it isn't as easy to show that this expression is greater than 1]

(4*) Show that $e^3 > 4e^{\frac{3}{2}}$

Solution

An equivalent result to prove is $e^{\frac{3}{2}} > 4$ (dividing both sides by $e^{\frac{3}{2}}$, which is positive) [you can never be sure what counts as being obvious]

$\Leftrightarrow e^3 > 16$ (as the function $y = x^2$ is increasing for $x > 0$)

$e^3 > (2 + 0.7)^3 > 2^3 + 3(2^2)(0.7) = 8 + 8.4 > 16$,

so that the original result is also true

(5*) Are the following true or false?

(i) $a < b \Rightarrow \frac{1}{a} > \frac{1}{b}$

(ii) $a < b \Rightarrow a^2 < b^2$

(iii) $a < b \;\&\; c < d \Rightarrow a + c < b + d$

(iv) $a < b \;\&\; c < d \Rightarrow a - c < b - d$

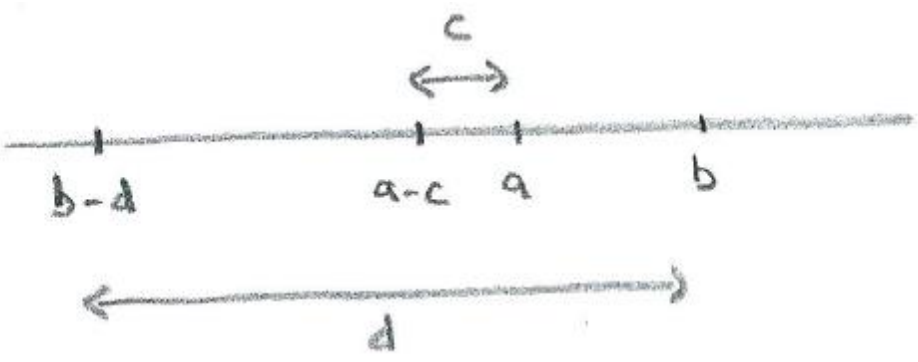
Solution

(i) Not true if $a < 0 \;\&\; b > 0$ (consider the graph of $y = 1/x$)

(ii) Not true if $a < 0 \;\&\; b < 0$ or if $a < 0, b > 0$ & $|b| < |a|$(consider the graph of $y = x^2$)

(iii) True: $a < b \Rightarrow a + c < b + c < b + d$

(iv) False: For example, $8 < 9$ and $2 < 4$, but it is not true that $8 - 2 < 9 - 4$; see diagram



(6*) Prove or provide a counter-example for the conjecture

$x > a$ & $y > b \Rightarrow xy > ab$ $\quad (a, b\ real)$ in each of the following cases:

(i) $a > 0, b > 0$ (ii) $a < 0, b < 0$ (iii) $a > 0, b < 0$

Solution

(i) $x > a \Rightarrow xy > ay$ [as $y > 0$] $> ab$ [since $y > b \Rightarrow ay > ab$]

so true

[or refer to graph of $y = ab$]

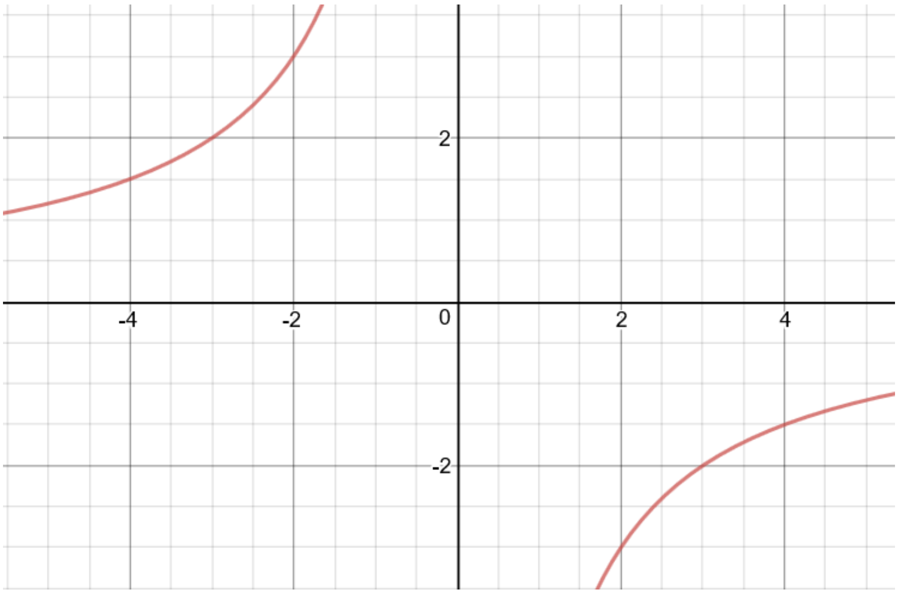
(b) $a < 0, b < 0$

counter-example: $x = 0$

(c) $a > 0, b < 0$

consider graph of $xy = ab$ when $a = 3, b = -2$ (see below)

(counter-example: $x = 4 + \delta, y = -2 + \delta$)



(7*) Prove that $a + b < 1 + ab$ if $a > 1$ and $b > 1$

Solution

$\Leftrightarrow a + b - 1 - ab < 0$

$\Leftrightarrow a(1 - b) - (1 - b) < 0$

$\Leftrightarrow (a - 1)(1 - b) < 0$

(8***) Solve the following inequality $\frac{x}{x-1} \leq \frac{3}{x+2}$ $(x \neq 1, x \neq -2)$

Solution

Method 1

$\frac{x}{x-1} \leq \frac{3}{x+2}$

Multiply both sides by $(x - 1)^2(x + 2)^2$ [as this will be positive]:

$x(x - 1)(x + 2)^2 \leq 3(x - 1)^2(x + 2)$

$\Rightarrow (x - 1)(x + 2)\{x(x + 2) - 3(x - 1)\} \leq 0$

$\Rightarrow (x-1)(x+2)(x^2-x+3) \leq 0$

As $x^2 - x + 3 = \left(x - \frac{1}{2}\right)^2 - \frac{1}{4} + 3 > 0$ for all x,

the original inequality is satisfied when $-2 \leq x \leq 1$ [for example, by considering the graph of $y = (x-1)(x+2)$].

Method 2

$\frac{x}{x-1} \leq \frac{3}{x+2} \Rightarrow \frac{x}{x-1} - \frac{3}{x+2} \leq 0$

$\Rightarrow \frac{x(x+2)-3(x-1)}{(x-1)(x+2)} \leq 0 \Rightarrow \frac{x^2-x+3}{(x-1)(x+2)} \leq 0$

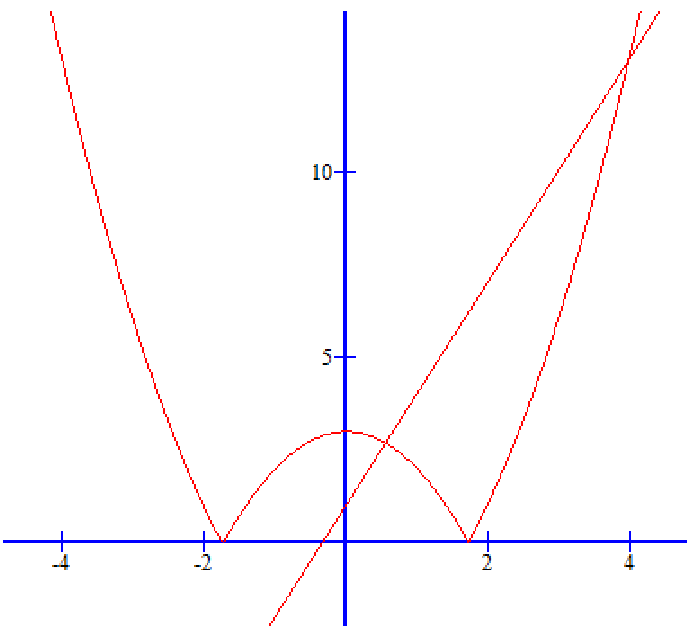
As before, $x^2 - x + 3 > 0$ for all x,

so that we require $(x-1)(x+2) \leq 0$, and hence $-2 \leq x \leq 1$

(9***) Solve the following inequality

$|x^2 - 3| > 3x + 1$

Solution



To determine the points of intersection of $y = |x^2 - 3|$ and $y = 3x + 1$:

$x^2 - 3 \geq 0$ and $x^2 - 3 = 3x + 1 \Rightarrow x^2 - 3x - 4 = 0 \Rightarrow (x - 4)(x + 1) = 0$

From the graph, we can reject the negative value of x, to give $x = 4$, which is consistent with $x^2 - 3 \geq 0$.

Also, $x^2 - 3 < 0$ and $-(x^2 - 3) = 3x + 1 \Rightarrow x^2 + 3x - 2 = 0$

$\Rightarrow x = \frac{-3 \pm \sqrt{9+8}}{2}$

From the graph, we can reject the negative value of x, to give $\frac{-3+\sqrt{17}}{2}$, which is consistent with $x^2 - 3 < 0$.

So, from the graph, $|x^2 - 3| > 3x + 1$ when $x < \frac{-3+\sqrt{17}}{2}$ or $x > 4$.