

## Induction (4 pages; 4/10/20)

### Example 1

$$\sum_{r=1}^n r^3 = \frac{1}{4}n^2(n+1)^2$$

### Solution

$\sum_{r=1}^1 r^3 = 1^3 = 1$  and  $\frac{1}{4}(1)^2(1+1)^2 = 1$ ; thus the result is true for  $n = 1$

[Be careful to give enough working for both sides.]

Now assume that the result is true for  $n = k$ , so that

$$\sum_{r=1}^k r^3 = \frac{1}{4}k^2(k+1)^2$$

[A common error is to write "Let  $n = k$ ", or "Assume  $n = k$ ", or "If  $n = k$ "]

[At this point it is possible to indicate the 'target' for  $n = k + 1$ ; namely that  $\sum_{r=1}^{k+1} r^3 = \frac{1}{4}(k+1)^2(k+2)^2$ ]

$$\begin{aligned} \text{Then } \sum_{r=1}^{k+1} r^3 &= \frac{1}{4}k^2(k+1)^2 + (k+1)^3 \\ &= \frac{1}{4}(k+1)^2(k^2 + 4(k+1)) \\ &= \frac{1}{4}(k+1)^2(k+2)^2 \\ &= \frac{1}{4}(k+1)^2([k+1] + 1)^2 \end{aligned}$$

But this is the result with  $k$  replaced by  $k + 1$ .

[Or, if the target has been mentioned previously, just indicate that the target has been obtained.]

So, if the result is true for  $n = k$ , then it is true for  $n = k + 1$ .

[A common error is to write: "So the result is true for  $n = k$  and  $n = k + 1$ "]

As the result is true for  $n = 1$ , it is therefore true for  $n = 2, 3, \dots$  and, by [the principle of mathematical] induction, for all integer  $n \geq 1$ . [In some cases, it is appropriate to start at a different value for  $n$ , such as 0 or 2. This depends on what values of  $n$  the given formula is defined for.]

[Note that no credit is ever given in an exam for the highlighted wording on its own, or where the algebra is 'fudged'.]

### Example 2

(1) If  $u_{n+1} = 2u_n + 3$ , where  $u_1 = 5$ , then  $u_n = 2^{n+2} - 3$

### Solution

[Show that the result is true for  $n = 1$ ]

Now assume that the result is true for  $n = k$ , so that

$$u_k = 2^{k+2} - 3$$

$$\begin{aligned} \text{Then } u_{k+1} &= 2u_k + 3 = 2(2^{k+2} - 3) + 3 = 2^{k+3} - 3 \\ &= 2^{(k+1)+2} - 3 \end{aligned}$$

[Standard wording]

### Example 3

(1)  $7^n + 4^n + 1$  is divisible by 6

### Solution

[Show that the result is true for  $n = 1$ ]

Now assume that the result is true for  $n = k$

### Approach 1

so that  $7^k + 4^k + 1 = 6M$ , where  $M \in \mathbb{Z}^+$

To show that the result is then true for  $n = k + 1$ :

$$7^{k+1} + 4^{k+1} + 1 = 7(7^k + 4^k + 1) - 3(4^k) - 6$$

$$= 7(6M) - 6(2)(4^{k-1}) - 6$$

$$= 6(7M - 2(4^{k-1}) - 1), \text{ which is a multiple of 6 for } k \geq 1$$

(the multiple is positive, as  $7^{k+1} + 4^{k+1} + 1$  is positive)

[Standard wording]

### Approach 2

Let  $f(k) = 7^k + 4^k + 1$

$$\text{Then } f(k+1) - \lambda f(k) = (7^{k+1} + 4^{k+1} + 1) - \lambda(7^k + 4^k + 1)$$

[an appropriate  $\lambda$  will be chosen shortly]

$$= 7^k(7 - \lambda) + 4^k(4 - \lambda) + 1 - \lambda$$

$$\text{Let } \lambda = 7, \text{ so that } f(k+1) - 7f(k) = -3(4^k) - 6$$

$$\text{and } f(k+1) = 7f(k) - 6(2)(4^{k-1}) - 6$$

As  $f(k)$  is assumed to be a multiple of 6, and the other terms on the RHS are also multiples of 6 (for  $k \geq 1$ ), it follows that

$f(k+1)$  is a multiple of 6 (the multiple is positive, as

$7^{k+1} + 4^{k+1} + 1$  is positive).

[Standard wording]

[Textbooks sometimes consider  $f(k + 1) - f(k)$  (ie with  $\lambda = 1$ ), but this isn't guaranteed to work. (See Induction Exercises for examples of this.)]