Induction - Exercises (Solutions) (23 pages; 30/1/18)

(1) The sum of the 1st n odd numbers is n^2

Solution

Result to prove: $1 + 3 + 5 + \cdots + (2n - 1) = n^2$

[Apparently this was the first published proof by induction.]

[The 1st step is often to rewrite the LHS using the summation sign.]

Result to prove: $\sum_{r=1}^{n}(2r - 1) = n^2$

[Show that the result is true for $n = 1$]

Now assume that the result is true for $n = k$, so that

$\sum_{r=1}^{k}(2r - 1) = k^2$

The target result is $\sum_{r=1}^{k+1}(2r - 1) = (k + 1)^2$

Then $\sum_{r=1}^{k+1}(2r - 1) = k^2 + (2[k + 1] - 1)$

$= k^2 + 2k + 1 = (k + 1)^2$, which is the target.

[Standard wording]

(2) $1 \times 4 + 2 \times 5 + 3 \times 6 + \cdots + n(n + 3) = \frac{1}{3}n(n + 1)(n + 5)$

Solution

Result to prove: $\sum_{r=1}^{n} r(r + 3) = \frac{1}{3}n(n + 1)(n + 5)$

[Show that the result is true for $n = 1$]

Now assume that the result is true for $n = k$, so that

$\sum_{r=1}^{k} r(r+3) = \frac{1}{3}k(k+1)(k+5)$

The target result is

$\sum_{r=1}^{k+1} r(r+3) = \frac{1}{3}(k+1)(k+2)(k+6)$

Then $\sum_{r=1}^{k+1} r(r+3) = \frac{1}{3}k(k+1)(k+5) + (k+1)(k+4)$

$= \frac{1}{3}(k+1)\{k(k+5) + 3(k+4)\} = \frac{1}{3}(k+1)(k^2+8k+12)$

$= \frac{1}{3}(k+1)(k+2)(k+6)$, which is the target.

[Standard wording]

(3) $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} = 1 - \frac{1}{2^n}$

Solution

Result to prove: $\sum_{r=1}^{n} \frac{1}{2^r} = 1 - \frac{1}{2^n}$

[Show that the result is true for $n = 1$]

Now assume that the result is true for $n = k$, so that

$\sum_{r=1}^{k} \frac{1}{2^r} = 1 - \frac{1}{2^k}$

The target result is $\sum_{r=1}^{k+1} \frac{1}{2^r} = 1 - \frac{1}{2^{k+1}}$

Then $\sum_{r=1}^{k+1} \frac{1}{2^r} = 1 - \frac{1}{2^k} + \frac{1}{2^{k+1}}$

$= 1 - \frac{1}{2^{k+1}}(2-1) = 1 - \frac{1}{2^{k+1}}$, which is the target.

[Standard wording]

(4) $2+4+6+\cdots+2n=n(n+1)$

Solution

Result to prove: $\sum_{r=1}^{n} 2r = n(n+1)$

[Show that the result is true for $n=1$]

Now assume that the result is true for $n=k$, so that

$\sum_{r=1}^{k} 2r = k(k+1)$

Then $\sum_{r=1}^{k+1} 2r = k(k+1)+2(k+1)=(k+1)(k+2)$

$(k+1)([k+1]+1)$

[Standard wording]

(5) $\sum_{r=1}^{n} r(r+1) = \frac{1}{3}n(n+1)(n+2)$

Solution

[Show that the result is true for $n=1$]

Now assume that the result is true for $n=k$, so that

$\sum_{r=1}^{k} r(r+1) = \frac{1}{3}k(k+1)(k+2)$

Then $\sum_{r=1}^{k+1} r(r+1) = \frac{1}{3}k(k+1)(k+2)+(k+1)(k+2)$

$=\frac{1}{3}(k+1)(k+2)(k+3)$

$=\frac{1}{3}(k+1)([k+1]+1)([k+1]+2)$

[Standard wording]

(6) $\sum_{r=1}^{n} r(r+2) = \frac{1}{6}n(n+1)(2n+7)$

Solution

[Show that the result is true for $n = 1$]

Now assume that the result is true for $n = k$, so that

$\sum_{r=1}^{k} r(r+2) = \frac{1}{6}k(k+1)(2k+7)$

Then $\sum_{r=1}^{k+1} k(k+2) = \frac{1}{6}k(k+1)(2k+7) + (k+1)(k+3)$

$= \frac{1}{6}(k+1)\{k(2k+7) + 6(k+3)\}$

$= \frac{1}{6}(k+1)(2k^2 + 13k + 18)$

$= \frac{1}{6}(k+1)(2k+9)(k+2)$

$= \frac{1}{6}(k+1)([k+1]+1)(2[k+1]+7)$

[Standard wording]

(7) $\sum_{r=1}^{n} r(r+1)(r+2) = \frac{1}{4}n(n+1)(n+2)(n+3)$

Solution

[Show that the result is true for $n = 1$]

Now assume that the result is true for $n = k$, so that

$\sum_{r=1}^{k} r(r+1)(r+2) = \frac{1}{4}k(k+1)(k+2)(k+3)$

Then $\sum_{r=1}^{k+1} r(r+1)(r+2)$

$= \frac{1}{4}k(k+1)(k+2)(k+3) + (k+1)(k+2)(k+3)$

$= \frac{1}{4}(k+1)(k+2)(k+3)(k+4)$

$= \frac{1}{4}(k+1)([k+1]+1)([k+1]+2)([k+1]+3)$

[Standard wording]

(8) $\sum_{r=1}^{n} 2^r = 2(2^n - 1)$

Solution

[Show that the result is true for $n = 1$]

Now assume that the result is true for $n = k$, so that

$\sum_{r=1}^{k} 2^r = 2(2^k - 1)$

Then $\sum_{r=1}^{k+1} 2^r = 2(2^k - 1) + 2^{k+1}$

$= 2^{k+1}(1+1) - 2 = 2(2^{k+1} - 1)$

[Standard wording]

(9) $\sum_{r=1}^{n} \frac{1}{(2r-1)(2r+1)} = \frac{n}{2n+1}$

Solution

[Show that the result is true for $n = 1$]

Now assume that the result is true for $n = k$, so that

$\sum_{r=1}^{k} \frac{1}{(2r-1)(2r+1)} = \frac{k}{2k+1}$

The target result is $\sum_{r=1}^{k+1} \frac{1}{(2r-1)(2r+1)} = \frac{k+1}{2k+3}$

LHS $= \frac{k}{2k+1} + \frac{1}{(2k+1)(2k+3)}$

$= \frac{k(2k+3)+1}{(2k+1)(2k+3)} = \frac{2k^2+3k+1}{(2k+1)(2k+3)} = \frac{(2k+1)(k+1)}{(2k+1)(2k+3)} = \frac{k+1}{2k+3}$

[Standard wording]

(10) $\sum_{r=1}^{n}\frac{1}{r(r+1)(r+2)}=\frac{n(n+3)}{4(n+1)(n+2)}$

Solution

[Show that the result is true for $n=1$]

Now assume that the result is true for $n=k$, so that

$\sum_{r=1}^{k}\frac{1}{r(r+1)(r+2)}=\frac{k(k+3)}{4(k+1)(k+2)}$

The target result is $\sum_{r=1}^{k+1}\frac{1}{r(r+1)(r+2)}=\frac{(k+1)(k+4)}{4(k+2)(k+3)}$

LHS $=\frac{k(k+3)}{4(k+1)(k+2)}+\frac{1}{(k+1)(k+2)(k+3)}$

$=\frac{k(k+3)(k+3)+4}{4(k+1)(k+2)(k+3)}=\frac{k^3+6k^2+9k+4}{4(k+1)(k+2)(k+3)}$

$=\frac{(k+1)(k^2+5k+4)}{4(k+1)(k+2)(k+3)}=\frac{(k+1)(k+4)}{4(k+2)(k+3)}$

[Standard wording]

(11) $\sum_{r=1}^{n}r(r!)=(n+1)!-1$

Solution

[Show that the result is true for $n=1$]

Now assume that the result is true for $n=k$, so that

$\sum_{r=1}^{k}r(r!)=(k+1)!-1$

Then $\sum_{r=1}^{k+1}r(r!)=(k+1)!-1+(k+1)(k+1)!$

$=(k+1)!\,(k+2)-1=(k+2)!-1$

$=([k+1]+1)!-1$

[Standard wording]

Type B

(1) If $u_n = u_{n-1} + 2$, where $u_1 = 3$, then $u_n = 2n + 1$

Solution

[Show that the result is true for $n = 1$]

Now assume that the result is true for $n = k$

so that $u_k = 2k + 1$

Then $u_{k+1} = u_k + 2 = 2k + 3 = 2(k + 1) + 1$

[Standard wording]

(2) If $u_n = 3u_{n-1} + 4$, where $u_1 = 2$, then $u_n = 4(3^{n-1}) - 2$

Solution

[Show that the result is true for $n = 1$]

Now assume that the result is true for $n = k$, so that

$u_k = 4(3^{k-1}) - 2$

Then $u_{k+1} = 3u_k + 4 = 12(3^{k-1}) - 6 + 4$

$= 4\left(3^{(k+1)-1}\right) - 2$

[Standard wording]

(3) If $u_n = 3u_{n-1} - 2u_{n-2}$, where $u_1 = 1$ & $u_2 = 3$,

then $u_n = 2^n - 1$

Solution

Assume that the result is true for $n = k$ and $n = k + 1$,

so that $u_k = 2^k - 1$ and $u_{k+1} = 2^{k+1} - 1$

Then $u_{k+2} = 3u_{k+1} - 2u_k = 3(2^{k+1} - 1) - 2(2^k - 1)$

$= 2^{k+1}(3 - 1) - 1 = 2^{k+2} - 1$, which is the required result for $n = k + 2$.

Thus if the result is true for $n = k$ and $n = k + 1$, then it is true for $n = k + 2$.

[Show true for $n = 1$ & $n = 2$]

Hence it is true for $n = 3, 4, \ldots$ etc

(4) If $u_n = 5u_{n-1} - 6u_{n-2}$, where $u_0 = -1$ & $u_1 = -1$,

then $u_n = 3^n - 2^{n+1}$

Solution

[Show that the result is true for $n = 1$]

[We start with $n = 1$ because then the method is reliant on using the expression for u_{k+1} , where $k = 1$, and this is defined (as

$5u_k - 6u_{k-1} = 5u_1 - 6u_0$), whereas the corresponding expression for $k = 0$ ($5u_0 - 6u_{-1}$) is not defined.]

Now assume that the result is true for $n = k$,

so that $u_k = 3^k - 2^{k+1}$

Then $u_{k+1} = 5u_k - 6u_{k-1} = 5(3^k - 2^{k+1}) - 6(3^{k-1} - 2^k)$

$= 3^{k-1}(15 - 6) - 2^k(10 - 6)$

$= 3^{k+1} - 2^{k+2}$

$= 3^{k+1} - 2^{(k+1)+1}$

[Standard wording]

(5) If $u_{n+1} = 3u_n - 2^n$, where $u_1 = 5$, then $u_n = 2^n + 3^n$

Solution

[Show that the result is true for $n = 1$]

Now assume that the result is true for $n = k$,

so that $u_k = 2^k + 3^k$

Then $u_{k+1} = 3u_k - 2^k = 3(2^k + 3^k) - 2^k$

$= (3 - 1)2^k + 3^{k+1}$

[this avoids writing down the last line straightaway - as it's effectively a 'show that' result]

$= 2^{k+1} + 3^{k+1}$

[Standard wording]

(6) If $u_{n+1} = 4n - u_n$, where $u_1 = \frac{1}{2}$,

then $u_n = 2n + \frac{1}{2}(-1)^n - 1$

Solution

[Show that the result is true for $n = 1$]

Now assume that the result is true for $n = k$,

so that $u_k = 2k + \frac{1}{2}(-1)^k - 1$

Then $u_{k+1} = 4k - (2k + \frac{1}{2}(-1)^k - 1)$

$= 2k + \frac{1}{2}(-1)^{k+1} + 1$

$= 2(k+1) + \frac{1}{2}(-1)^{k+1} - 1$

[Standard wording]

(7) If $u_{n+1} = \frac{u_n}{u_n+1}$, where $u_n = 1$, suggest a formula for u_n and prove it by induction

Solution

$u_2 = \frac{u_1}{u_1+1} = \frac{1}{2}$, $u_3 = \frac{u_2}{u_2+1} = \frac{\frac{1}{2}}{\frac{1}{2}+1} = \frac{1}{3}$, $u_4 = \frac{u_3}{u_3+1} = \frac{\frac{1}{3}}{\frac{1}{3}+1} = \frac{1}{4}$

Suppose that $u_n = \frac{1}{n}$

[Show that the result is true for $n = 1$]

Now assume that the result is true for $n = k$,

so that $u_k = \frac{1}{k}$

Then $u_{k+1} = \frac{u_k}{u_k+1} = \frac{\frac{1}{k}}{\frac{1}{k}+1} = \frac{1}{k+1}$

[Standard wording]

Type C

(1) $7^{2n-1} + 3^{2n}$ is divisible by 8

Solution

[Show that the result is true for $n = 1$]

Now assume that the result is true for $n = k$

Approach 1

so that $7^{2k-1} + 3^{2k} = 8M$, where $M \in \mathbb{Z}^+$

To show that the result is then true for $n = k + 1$:

$7^{2(k+1)-1} + 3^{2(k+1)} = 7^{2k+1} + 3^{2k+2}$

$= 49(7^{2k-1}) + 3^{2k+2}$

$= 49(8M - 3^{2k}) + 3^{2k+2}$

$= 8(49M) + 3^{2k}(-49 + 9)$

$= 8(49M - 5(3^{2k}))$

(the multiple is positive, as $7^{2(k+1)-1} + 3^{2(k+1)}$ is positive)

[Standard wording]

Approach 2

Let $f(k) = 7^{2k-1} + 3^{2k}$

Then $f(k + 1) - \lambda f(k) = 7^{2k+1} + 3^{2k+2} - \lambda(7^{2k-1} + 3^{2k})$

$= 7^{2k-1}(49 - \lambda) + 3^{2k}(9 - \lambda)$

Let $\lambda = 1$, so that $f(k + 1) = f(k) + 48(7^{2k-1}) + 8(3^{2k})$

As $f(k)$ is assumed to be a multiple of 8, and the other terms on the RHS are also a multiple of 8, it follows that $f(k + 1)$ is a multiple of 8. [Standard wording]

(2) $2^{n+2} + 3^{2n+1}$ is divisible by 7

Solution

[Show that the result is true for $n = 1$]

Now assume that the result is true for $n = k$

Approach 1

so that $2^{k+2} + 3^{2k+1} = 7M$, where $M \in \mathbb{Z}^+$

To show that the result is then true for $n = k + 1$:

$2^{(k+1)+2} + 3^{2(k+1)+1} = 2^{k+3} + 3^{2k+3}$

$= 2(2^{k+2}) + 3^{2k+3} = 2(7M - 3^{2k+1}) + 3^{2k+3}$

$= 7(2M) + 3^{2k+1}(-2 + 9)$

$= 7(2M + 3^{2k+1})$

[Standard wording]

Approach 2

Let $f(k) = 2^{k+2} + 3^{2k+1}$

Then $f(k+1) - \lambda f(k) = 2^{k+3} + 3^{2k+3} - \lambda(2^{k+2} + 3^{2k+1})$

$= 2^{k+2}(2 - \lambda) + 3^{2k+1}(9 - \lambda)$

Let $\lambda = 2$, so that $f(k+1) = 2f(k) + 7(3^{2k+1})$

As both terms on the RHS are multiples of 7, it follows that $f(k+1)$ is a multiple of 7.

[Note that we cannot set $\lambda = 1$ in this example.]

[Standard wording]

(3) $5^n + 12n - 1$ is divisible by 16

Solution

[Show that the result is true for $n = 1$]

Now assume that the result is true for $n = k$

Approach 1

so that $5^k + 12k - 1 = 16M$, where $M \in \mathbb{Z}^+$

To show that the result is then true for $n = k + 1$:

$5^{k+1} + 12(k + 1) - 1 = 5(16M - 12k + 1) + 12k + 11$

$= 16(5M) - 48k + 16$

$= 16(5M - 3k + 1)$

(the multiple is positive, as $5^{k+1} + 12(k + 1) - 1$ is positive)

[Standard wording]

Approach 2

Let $f(k) = 5^k + 12k - 1$

Then $f(k + 1) - \lambda f(k)$

$= 5^{k+1} + 12(k + 1) - 1 - \lambda(5^k + 12k - 1)$

$= 5^k(5 - \lambda) + 12k(1 - \lambda) + 11 + \lambda$

Let $\lambda = 5$, so that $f(k + 1) = 5f(k) - 48k + 16$

As all the terms on the RHS are multiples of 16, it follows that $f(k + 1)$ is a multiple of 16.

[Standard wording]

(4) $2^{n+1} + 9(13^n)$ is divisible by 11

Solution

[Show that the result is true for $n = 1$]

Now assume that the result is true for $n = k$

Approach 1

so that $2^{k+1} + 9(13^k) = 11M$, where $M \in \mathbb{Z}^+$

To show that the result is then true for $n = k + 1$:

$2^{k+2} + 9(13^{k+1}) = 2\{11M - 9(13^k)\} + 9(13^{k+1})$

$= 11(2M) + 13^k\{9(13) - 18)\}$

$= 11(2M) + 99(13^k)$

$= 11(2M + 9(13^k))$

[Standard wording]

Approach 2

Let $f(k) = 2^{k+1} + 9(13^k)$

Then $f(k+1) - \lambda f(k)$

$= 2^{k+2} + 9(13^{k+1}) - \lambda(2^{k+1} + 9(13^k))$

$= 2^{k+1}(2 - \lambda) + 13^k(117 - 9\lambda)$

Let $\lambda = 2$, so that $f(k+1) = 2f(k) + 99(13^k)$

As both terms on the RHS are multiples of 11, it follows that $f(k+1)$ is a multiple of 11.

[Standard wording]

(5) $13^n + 6^{n-1}$ is divisible by 7

Solution

[Show that the result is true for $n = 1$]

Now assume that the result is true for $n = k$

Approach 1

so that $13^k + 6^{k-1} = 7M$, where $M \in \mathbb{Z}^+$

To show that the result is then true for $n = k + 1$:

$13^{k+1} + 6^k = 13(7M - 6^{k-1}) + 6^k$

$= 7(13M) + 6^{k-1}(6 - 13)$

$= 7(13M - 6^{k-1})$

(the multiple is positive, as $13^{k+1} + 6^k$ is positive)

[Standard wording]

Approach 2

Let $f(k) = 13^k + 6^{k-1}$

Then $f(k+1) - \lambda f(k)$

$= (13^{k+1} + 6^k) - \lambda(13^k + 6^{k-1})$

$= 13^k(13 - \lambda) + 6^{k-1}(6 - \lambda)$

putting $\lambda = 13$ [or $\lambda = -1$]

$= -7(6^{k-1})$

so that $f(k+1) = 13f(k) - 7(6^{k-1})$

As both terms on the RHS are multiples of 7, it follows that $f(k+1)$ is a multiple of 7

(the multiple is positive, as $13^{k+1} + 6^k$ is positive)

[Standard wording]

(6) $5^{2n} + 12^{n-1}$ is divisible by 13

Solution

[Show that the result is true for $n = 1$]

Now assume that the result is true for $n = k$

Approach 1

so that $5^{2k} + 12^{k-1} = 13M$, where $M \in \mathbb{Z}^+$

To show that the result is then true for $n = k + 1$:

$5^{2k+2} + 12^k = 25(13M - 12^{k-1}) + 12^k$

$= 13(25M) + 12^{k-1}(12 - 25)$

$= 13(25M - 12^{k-1})$

(the multiple is positive, as $5^{2k+2} + 12^k$ is positive)

[Standard wording]

Approach 2

Let $f(k) = 5^{2k} + 12^{k-1}$

Then $f(k+1) - \lambda f(k)$

$= 5^{2k+2} + 12^k - \lambda(5^{2k} + 12^{k-1})$

$= 5^{2k}(25 - \lambda) + 12^{k-1}(12 - \lambda)$

putting $\lambda = 25$ [or $\lambda = -1$]

$= -(13)12^{k-1}$

so that $f(k+1) = 25f(k) - 13(12^{k-1})$

As both terms on the RHS are multiples of 13, it follows that $f(k+1)$ is a multiple of 13

(the multiple is positive, as $5^{2k+2} + 12^k$ is positive)

[Standard wording]

(7) $5^{2n+2} - 24n - 25$ is divisible by 576

Solution

[Show that the result is true for $n = 1$]

Now assume that the result is true for $n = k$

Approach 1

so that $5^{2k+2} - 24k - 25 = 576M$, where $M \in \mathbb{Z}^+$

To show that the result is then true for $n = k + 1$:

$5^{2k+4} - 24(k + 1) - 25 = 25(576M + 24k + 25) - 24k - 49$

$= 576(25M) + 24k(24) + 576$

$= 576(25M + k + 1)$

[Standard wording]

Approach 2

Let $f(k) = 5^{2k+2} - 24k - 25$

Then $f(k + 1) - \lambda f(k)$

$= 5^{2k+4} - 24(k + 1) - 25 - \lambda(5^{2k+2} - 24k - 25)$

$= 5^{2k+2}(25 - \lambda) + k(-24 + 24\lambda) - 49 + 25\lambda$

putting $\lambda = 25$

$= 24(24)k - 49 + 625 = 576(k + 1)$

so that $f(k + 1) = 25f(k) + 576(k + 1)$

As both terms on the RHS are multiples of 576, it follows that $f(k + 1)$ is a multiple of 576

[Standard wording]

(8) $2^{4n+1} + 3$ is divisible by 5

Solution

[Show that the result is true for $n = 1$]

Now assume that the result is true for $n = k$

Approach 1

so that $2^{4k+1} + 3 = 5M$, where $M \in \mathbb{Z}^+$

To show that the result is then true for $n = k + 1$:

$2^{4k+5} + 3 = 16(5M - 3) + 3$

$= 5(16M - 9)$

(the multiple is positive, as $2^{4k+5} + 3$ is positive)

[Standard wording]

Approach 2

Let $f(k) = 2^{4k+1} + 3$

Then $f(k + 1) - \lambda f(k)$

$= (2^{4k+5} + 3) - \lambda(2^{4k+1} + 3)$

$= 2^{4k+1}(16 - \lambda) + 3(1 - \lambda)$

putting $\lambda = 16$ [or $\lambda = 1$]

$= -45$

so that $f(k + 1) = 16f(k) - 45$

As both terms on the RHS are multiples of 5, it follows that $f(k + 1)$ is a multiple of 5 (and the multiple is positive, as

$2^{4k+5} + 3$ is positive)

[Standard wording]

Type D

(1) $2 + 4 + 6 + \cdots + 2n > n^2$

Solution

Result to prove: $2\sum_{r=1}^{n} r > n^2$

[Show that the result is true for $n = 1$]

Now assume that the result is true for $n = k$

so that $2\sum_{r=1}^{k} r > k^2$

Then $2\sum_{r=1}^{k+1} r > k^2 + 2(k+1)$

$= (k+1)^2 + 1 > (k+1)^2$

[Standard wording]

(2) $\sum_{r=1}^{n} r^2 > \frac{1}{3}n^3$

Solution

[Show that the result is true for $n = 1$]

Now assume that the result is true for $n = k$

so that $\sum_{r=1}^{k} r^2 > \frac{1}{3}k^3$

Then $\sum_{r=1}^{k+1} r^2 > \frac{1}{3}k^3 + (k+1)^2$

$= \frac{1}{3}(k+1)^3 - \frac{1}{3}(3k^2 + 3k + 1) + (k+1)^2$

$= \frac{1}{3}(k+1)^3 + \frac{1}{3}(-3k + 1 + 6k + 3)$

$= \frac{1}{3}(k+1)^3 + \frac{1}{3}(3k + 4)$

$> \frac{1}{3}(k+1)^3$

[Standard wording]

(3) $\frac{1}{4}n^4 < \sum_{r=1}^{n} r^3 \leq n^4$

Solution

[Show that the result is true for $n = 1$]

Now assume that the result is true for $n = k$

so that $\frac{1}{4}k^4 < \sum_{r=1}^{k} r^3 \leq k^4$

Then $\sum_{r=1}^{k+1} r^3 > \frac{1}{4}k^4 + (k+1)^3$

$= \frac{1}{4}(k+1)^4 - \frac{1}{4}(4k^3 + 6k^2 + 4k + 1) + (k+1)^3$

$= \frac{1}{4}(k+1)^4 - \frac{1}{4}(6k^2 + 4k + 1) + 3k^2 + 3k + 1$

$= \frac{1}{4}(k+1)^4 + \frac{3}{2}k^2 + 2k + \frac{3}{4}$

$> \frac{1}{4}(k+1)^4$ (as $k > 0$)

Also, $\sum_{r=1}^{k+1} r^3 \leq k^4 + (k+1)^3$

$= (k+1)^4 - 4k^3 - 6k^2 - 4k - 1 + (k+1)^3$

$= (k+1)^4 - 4k^3 - 6k^2 - 4k - 1 + k^3 + 3k^2 + 3k + 1$

$= (k+1)^4 - 3k^3 - 3k^2 - k$

$< (k+1)^4$ (as $k > 0$)

$\leq (k+1)^4$

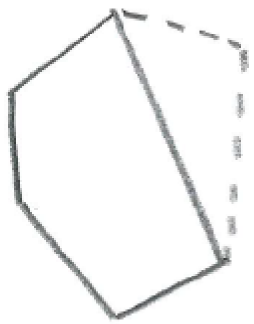
[Standard wording]

(4) The sum of the interior angles of a convex n-sided polygon is $180(n-2)$

Solution

When $n = 3$ (the smallest possible value), the result is true, as the interior angles of a triangle add up to 180°.

Now assume that the result is true for $n = k$, so that the total of the interior angles is $180(k-2)$.



The diagram shows the case $k = 5$, but applies more generally.

By adding another triangle, n has increased by 1, and the total of the interior angles has increased by 180.

Thus the total for $k+1$ sides is $180(k-2) + 180 = 180(k-1)$

$= 180([k+1]-2)$

[Standard wording, but starting at $n = 3$]

(5) If $A = \begin{pmatrix} -1 & -4 \\ 1 & 3 \end{pmatrix}$, then $A^n = \begin{pmatrix} 1-2n & -4n \\ n & 1+2n \end{pmatrix}$

Solution

[Show that the result is true for $n = 1$]

Now assume that the result is true for $n = k$

so that $A^k = \begin{pmatrix} 1-2k & -4k \\ k & 1+2k \end{pmatrix}$

Target is: $A^{k+1} = \begin{pmatrix} -1-2k & -4k-4 \\ k+1 & 3+2k \end{pmatrix}$

$A^{k+1} = \begin{pmatrix} -1 & -4 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1-2k & -4k \\ k & 1+2k \end{pmatrix}$

$= \begin{pmatrix} -1+2k-4k & 4k-4-8k \\ 1-2k+3k & -4k+3+6k \end{pmatrix}$

$= \begin{pmatrix} -1-2k & -4k-4 \\ k+1 & 3+2k \end{pmatrix}$

[Standard wording]

(6) $\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\dots\left(1-\frac{1}{n^2}\right) = \frac{n+1}{2n}$ for $n \geq 2$

Solution

[Show that the result is true for $n = 2$]

Now assume that the result is true for $n = k$

so that $\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\dots\left(1-\frac{1}{k^2}\right) = \frac{k+1}{2k}$

Target: $\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\dots\left(1-\frac{1}{(k+1)^2}\right) = \frac{k+2}{2k+2}$

LHS $= \left(\frac{k+1}{2k}\right)\left(1-\frac{1}{(k+1)^2}\right) = \left(\frac{k+1}{2k}\right)\left(\frac{(k+1)^2-1}{(k+1)^2}\right)$

$= \left(\frac{1}{2k}\right)\left(\frac{k^2+2k}{k+1}\right) = \frac{k(k+2)}{k(2k+2)} = \frac{k+2}{2k+2}$

[Standard wording, but starting at $n = 2$]

(7) (i) If $y = e^x sinx$, show that $\frac{dy}{dx} = \sqrt{2}\, e^x \sin(x+\frac{\pi}{4})$

(ii) Show that $\frac{d^n y}{dx^n} = \left(\sqrt{2}\right)^n e^x \sin(x + \frac{n\pi}{4})$

Solution

(i) $\frac{dy}{dx} = e^x sinx + e^x cosx = \sqrt{2}\, e^x\{sinx\left(\frac{1}{\sqrt{2}}\right) + cosx\left(\frac{1}{\sqrt{2}}\right)\}$

$= \sqrt{2}\, e^x\{sinxcos\left(\frac{\pi}{4}\right) + cosxsin\left(\frac{\pi}{4}\right)\}$

$= \sqrt{2}\, e^x \sin(x + \frac{\pi}{4})$

(ii) [Show that the result is true for $n = 1$]

Now assume that the result is true for $n = k$,

so that $\frac{d^k y}{dx^k} = \left(\sqrt{2}\right)^k e^x \sin(x + \frac{k\pi}{4})$

Then $\frac{d^{k+1} y}{dx^{k+1}} = \left(\sqrt{2}\right)^k e^x \sin\left(x + \frac{k\pi}{4}\right) + \left(\sqrt{2}\right)^k e^x \cos(x + \frac{k\pi}{4})$

$= \left(\sqrt{2}\right)^{k+1} e^x \{\sin\left(x + \frac{k\pi}{4}\right)\left(\frac{1}{\sqrt{2}}\right) + \cos(x + \frac{k\pi}{4})\left(\frac{1}{\sqrt{2}}\right)\}$

$= \left(\sqrt{2}\right)^{k+1} e^x \{\sin\left(x + \frac{k\pi}{4}\right) cos\left(\frac{\pi}{4}\right) + \cos(x + \frac{k\pi}{4}) sin\left(\frac{\pi}{4}\right)\}$

$= \left(\sqrt{2}\right)^{k+1} e^x \sin(\left[x + \frac{k\pi}{4}\right] + \frac{\pi}{4})$

$= \left(\sqrt{2}\right)^{k+1} e^x \sin(x + \frac{(k+1)\pi}{4})$

[Standard wording]