Induction – Exercises: Series (22 pages ; 27/7/25)

The sum of the 1st n odd numbers is n^2

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Solution

Result to prove: $1 + 3 + 5 + \cdots + (2n - 1) = n^2$

[Apparently this was the first published proof by induction.]

[The 1st step is often to rewrite the LHS using the summation sign.]

Result to prove: $\sum_{r=1}^{n}(2r - 1) = n^2$

[Show that the result is true for $n = 1$]

Now assume that the result is true for $n = k$, so that

$\sum_{r=1}^{k}(2r - 1) = k^2$

The target result is $\sum_{r=1}^{k+1}(2r - 1) = (k + 1)^2$

Then $\sum_{r=1}^{k+1}(2r - 1) = k^2 + (2[k + 1] - 1)$

$= k^2 + 2k + 1 = (k + 1)^2$, which is the target.

[Standard wording]

$$1 \times 4 + 2 \times 5 + 3 \times 6 + \cdots + n(n+3) = \frac{1}{3}n(n+1)(n+5)$$

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Solution

Result to prove: $\sum_{r=1}^{n} r(r+3) = \frac{1}{3}n(n+1)(n+5)$

[Show that the result is true for $n = 1$]

Now assume that the result is true for $n = k$, so that

$\sum_{r=1}^{k} r(r+3) = \frac{1}{3}k(k+1)(k+5)$

The target result is

$\sum_{r=1}^{k+1} r(r+3) = \frac{1}{3}(k+1)(k+2)(k+6)$

Then $\sum_{r=1}^{k+1} r(r+3) = \frac{1}{3}k(k+1)(k+5) + (k+1)(k+4)$

$= \frac{1}{3}(k+1)\{k(k+5) + 3(k+4)\} = \frac{1}{3}(k+1)(k^2 + 8k + 12)$

$= \frac{1}{3}(k+1)(k+2)(k+6)$, which is the target.

[Standard wording]

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+\frac{1}{2^n}=1-\frac{1}{2^n}$$

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Solution

Result to prove: $\sum_{r=1}^{n}\frac{1}{2^r}=1-\frac{1}{2^n}$

[Show that the result is true for $n=1$]

Now assume that the result is true for $n=k$, so that

$\sum_{r=1}^{k}\frac{1}{2^r}=1-\frac{1}{2^k}$

The target result is $\sum_{r=1}^{k+1}\frac{1}{2^r}=1-\frac{1}{2^{k+1}}$

Then $\sum_{r=1}^{k+1}\frac{1}{2^r}=1-\frac{1}{2^k}+\frac{1}{2^{k+1}}$

$=1-\frac{1}{2^{k+1}}(2-1)=1-\frac{1}{2^{k+1}}$, which is the target.

[Standard wording]

$$2 + 4 + 6 + \cdots + 2n = n(n + 1)$$

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Solution

Result to prove: $\sum_{r=1}^{n} 2r = n(n+1)$

[Show that the result is true for $n = 1$]

Now assume that the result is true for $n = k$, so that

$\sum_{r=1}^{k} 2r = k(k+1)$

Then $\sum_{r=1}^{k+1} 2r = k(k+1) + 2(k+1) = (k+1)(k+2)$

$(k+1)([k+1]+1)$

[Standard wording]

$\sum_{r=1}^{n} r(r+1) = \frac{1}{3}n(n+1)(n+2)$

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Solution

[Show that the result is true for $n = 1$]

Now assume that the result is true for $n = k$, so that

$\sum_{r=1}^{k} r(r+1) = \frac{1}{3}k(k+1)(k+2)$

Then $\sum_{r=1}^{k+1} r(r+1) = \frac{1}{3}k(k+1)(k+2) + (k+1)(k+2)$

$= \frac{1}{3}(k+1)(k+2)(k+3)$

$= \frac{1}{3}(k+1)([k+1]+1)([k+1]+2)$

[Standard wording]

$\sum_{r=1}^{n} r(r+2) = \frac{1}{6}n(n+1)(2n+7)$

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Solution

[Show that the result is true for $n = 1$]

Now assume that the result is true for $n = k$, so that

$\sum_{r=1}^{k} r(r+2) = \frac{1}{6}k(k+1)(2k+7)$

Then $\sum_{r=1}^{k+1} k(k+2) = \frac{1}{6}k(k+1)(2k+7) + (k+1)(k+3)$

$= \frac{1}{6}(k+1)\{k(2k+7) + 6(k+3)\}$

$= \frac{1}{6}(k+1)(2k^2 + 13k + 18)$

$= \frac{1}{6}(k+1)(2k+9)(k+2)$

$= \frac{1}{6}(k+1)([k+1]+1)(2[k+1]+7)$

[Standard wording]

$\sum_{r=1}^{n} r(r+1)(r+2) = \frac{1}{4}n(n+1)(n+2)(n+3)$

$\sum_{r=1}^{n} r(r+1)(r+2) = \frac{1}{4}n(n+1)(n+2)(n+3)$

Solution

[Show that the result is true for $n = 1$]

Now assume that the result is true for $n = k$, so that

$\sum_{r=1}^{k} r(r+1)(r+2) = \frac{1}{4}k(k+1)(k+2)(k+3)$

Then $\sum_{r=1}^{k+1} r(r+1)(r+2)$

$= \frac{1}{4}k(k+1)(k+2)(k+3) + (k+1)(k+2)(k+3)$

$= \frac{1}{4}(k+1)(k+2)(k+3)(k+4)$

$= \frac{1}{4}(k+1)([k+1]+1)([k+1]+2)([k+1]+3)$

[Standard wording]

$\sum_{r=1}^{n} 2^r = 2(2^n - 1)$

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Solution

[Show that the result is true for $n = 1$]

Now assume that the result is true for $n = k$, so that

$\sum_{r=1}^{k} 2^r = 2(2^k - 1)$

Then $\sum_{r=1}^{k+1} 2^r = 2(2^k - 1) + 2^{k+1}$

$= 2^{k+1}(1 + 1) - 2 = 2(2^{k+1} - 1)$

[Standard wording]

$\sum_{r=1}^{n} \frac{1}{(2r-1)(2r+1)} = \frac{n}{2n+1}$

$\sum_{r=1}^{n} \frac{1}{(2r-1)(2r+1)} = \frac{n}{2n+1}$

Solution

[Show that the result is true for $n = 1$]

Now assume that the result is true for $n = k$, so that

$\sum_{r=1}^{k} \frac{1}{(2r-1)(2r+1)} = \frac{k}{2k+1}$

The target result is $\sum_{r=1}^{k+1} \frac{1}{(2r-1)(2r+1)} = \frac{k+1}{2k+3}$

$\text{LHS} = \frac{k}{2k+1} + \frac{1}{(2k+1)(2k+3)}$

$= \frac{k(2k+3)+1}{(2k+1)(2k+3)} = \frac{2k^2+3k+1}{(2k+1)(2k+3)} = \frac{(2k+1)(k+1)}{(2k+1)(2k+3)} = \frac{k+1}{2k+3}$

[Standard wording]

$$\sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)} = \frac{n(n+3)}{4(n+1)(n+2)}$$

$\sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)} = \frac{n(n+3)}{4(n+1)(n+2)}$

Solution

[Show that the result is true for $n = 1$]

Now assume that the result is true for $n = k$, so that

$\sum_{r=1}^{k} \frac{1}{r(r+1)(r+2)} = \frac{k(k+3)}{4(k+1)(k+2)}$

The target result is $\sum_{r=1}^{k+1} \frac{1}{r(r+1)(r+2)} = \frac{(k+1)(k+4)}{4(k+2)(k+3)}$

$\text{LHS} = \frac{k(k+3)}{4(k+1)(k+2)} + \frac{1}{(k+1)(k+2)(k+3)}$

$= \frac{k(k+3)(k+3)+4}{4(k+1)(k+2)(k+3)} = \frac{k^3+6k^2+9k+4}{4(k+1)(k+2)(k+3)}$

$= \frac{(k+1)(k^2+5k+4)}{4(k+1)(k+2)(k+3)} = \frac{(k+1)(k+4)}{4(k+2)(k+3)}$

[Standard wording]

$\sum_{r=1}^{n} r(r!) = (n+1)! - 1$

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Solution

[Show that the result is true for $n = 1$]

Now assume that the result is true for $n = k$, so that

$\sum_{r=1}^{k} r(r!) = (k+1)! - 1$

Then $\sum_{r=1}^{k+1} r(r!) = (k+1)! - 1 + (k+1)(k+1)!$

$= (k+1)!\,(k+2) - 1 = (k+2)! - 1$

$= ([k+1]+1)! - 1$

[Standard wording]