Induction – Exercises: Sequences (14 pages ; 30/7/25)

If $u_n = u_{n-1} + 2$, where $u_1 = 3$, then $u_n = 2n + 1$

If $u_n = u_{n-1} + 2$, where $u_1 = 3$, then $u_n = 2n + 1$

Solution

[Show that the result is true for $n = 1$]

Now assume that the result is true for $n = k$

so that $u_k = 2k + 1$

Then $u_{k+1} = u_k + 2 = 2k + 3 = 2(k + 1) + 1$

[Standard wording]

If $u_n = 3u_{n-1} + 4$, where $u_1 = 2$, then $u_n = 4(3^{n-1}) - 2$

If $u_n = 3u_{n-1} + 4$, where $u_1 = 2$, then $u_n = 4(3^{n-1}) - 2$

Solution

[Show that the result is true for $n = 1$]

Now assume that the result is true for $n = k$, so that

$u_k = 4(3^{k-1}) - 2$

Then $u_{k+1} = 3u_k + 4 = 12(3^{k-1}) - 6 + 4$

$= 4\left(3^{(k+1)-1}\right) - 2$

[Standard wording]

If $u_n = 3u_{n-1} - 2u_{n-2}$, where $u_1 = 1$ & $u_2 = 3$,

then $u_n = 2^n - 1$

If $u_n = 3u_{n-1} - 2u_{n-2}$, where $u_1 = 1$ & $u_2 = 3$,

then $u_n = 2^n - 1$

Solution

Assume that the result is true for $n = k$ and $n = k + 1$,

so that $u_k = 2^k - 1$ and $u_{k+1} = 2^{k+1} - 1$

Then $u_{k+2} = 3u_{k+1} - 2u_k = 3(2^{k+1} - 1) - 2(2^k - 1)$

$= 2^{k+1}(3 - 1) - 1 = 2^{k+2} - 1$, which is the required result for $n = k + 2$.

Thus if the result is true for $n = k$ and $n = k + 1$, then it is true for $n = k + 2$.

[Show true for $n = 1$ & $n = 2$]

Hence it is true for $n = 3, 4, \ldots$ etc

If $u_n = 5u_{n-1} - 6u_{n-2}$, where $u_0 = -1$ & $u_1 = -1$,

then $u_n = 3^n - 2^{n+1}$

If $u_n = 5u_{n-1} - 6u_{n-2}$, where $u_0 = -1$ & $u_1 = -1$,

then $u_n = 3^n - 2^{n+1}$

Solution

[Show that the result is true for $n = 1$]

[We start with $n = 1$ because then the method is reliant on using the expression for u_{k+1} , where $k = 1$, and this is defined (as

$5u_k - 6u_{k-1} = 5u_1 - 6u_0$), whereas the corresponding expression for $k = 0$ ($5u_0 - 6u_{-1}$) is not defined.]

Now assume that the result is true for $n = k$,

so that $u_k = 3^k - 2^{k+1}$

Then $u_{k+1} = 5u_k - 6u_{k-1} = 5(3^k - 2^{k+1}) - 6(3^{k-1} - 2^k)$

$= 3^{k-1}(15 - 6) - 2^k(10 - 6)$

$= 3^{k+1} - 2^{k+2}$

$= 3^{k+1} - 2^{(k+1)+1}$

[Standard wording]

If $u_{n+1} = 3u_n - 2^n$, where $u_1 = 5$, then $u_n = 2^n + 3^n$

If $u_{n+1} = 3u_n - 2^n$, where $u_1 = 5$, then $u_n = 2^n + 3^n$

Solution

[Show that the result is true for $n = 1$]

Now assume that the result is true for $n = k$,

so that $u_k = 2^k + 3^k$

Then $u_{k+1} = 3u_k - 2^k = 3(2^k + 3^k) - 2^k$

$= (3 - 1)2^k + 3^{k+1}$

[this avoids writing down the last line straightaway - as it's effectively a 'show that' result]

$= 2^{k+1} + 3^{k+1}$

[Standard wording]

If $u_{n+1} = 4n - u_n$, where $u_1 = \frac{1}{2}$, then $u_n = 2n + \frac{1}{2}(-1)^n - 1$

If $u_{n+1} = 4n - u_n$, where $u_1 = \frac{1}{2}$, then $u_n = 2n + \frac{1}{2}(-1)^n - 1$

Solution

[Show that the result is true for $n = 1$]

Now assume that the result is true for $n = k$,

so that $u_k = 2k + \frac{1}{2}(-1)^k - 1$

Then $u_{k+1} = 4k - (2k + \frac{1}{2}(-1)^k - 1)$

$= 2k + \frac{1}{2}(-1)^{k+1} + 1$

$= 2(k + 1) + \frac{1}{2}(-1)^{k+1} - 1$

[Standard wording]

If $u_{n+1} = \frac{u_n}{u_n+1}$, where $u_n = 1$, suggest a formula for u_n and prove it by induction

If $u_{n+1} = \frac{u_n}{u_n+1}$, where $u_n = 1$, suggest a formula for u_n and prove it by induction

Solution

$u_2 = \frac{u_1}{u_1+1} = \frac{1}{2}$, $u_3 = \frac{u_2}{u_2+1} = \frac{\frac{1}{2}}{\frac{1}{2}+1} = \frac{1}{3}$, $u_4 = \frac{u_3}{u_3+1} = \frac{\frac{1}{3}}{\frac{1}{3}+1} = \frac{1}{4}$

Suppose that $u_n = \frac{1}{n}$

[Show that the result is true for $n = 1$]

Now assume that the result is true for $n = k$,

so that $u_k = \frac{1}{k}$

Then $u_{k+1} = \frac{u_k}{u_k+1} = \frac{\frac{1}{k}}{\frac{1}{k}+1} = \frac{1}{k+1}$

[Standard wording]