Induction – Exercises: Miscellaneous (12 pages ; 31/7/25)

$2 + 4 + 6 + \cdots + 2n > n^2$

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Solution

Result to prove: $2\sum_{r=1}^{n} r > n^2$

[Show that the result is true for $n = 1$]

Now assume that the result is true for $n = k$

so that $2\sum_{r=1}^{k} r > k^2$

Then $2\sum_{r=1}^{k+1} r > k^2 + 2(k+1)$

$= (k+1)^2 + 1 > (k+1)^2$

[Standard wording]

$\sum_{r=1}^{n} r^2 > \frac{1}{3} n^3$

$\sum_{r=1}^{n} r^2 > \frac{1}{3}n^3$

Solution

[Show that the result is true for $n = 1$]

Now assume that the result is true for $n = k$

so that $\sum_{r=1}^{k} r^2 > \frac{1}{3}k^3$

Then $\sum_{r=1}^{k+1} r^2 > \frac{1}{3}k^3 + (k+1)^2$

$= \frac{1}{3}(k+1)^3 - \frac{1}{3}(3k^2 + 3k + 1) + (k+1)^2$

$= \frac{1}{3}(k+1)^3 + \frac{1}{3}(-3k + 1 + 6k + 3)$

$= \frac{1}{3}(k+1)^3 + \frac{1}{3}(3k + 4)$

$> \frac{1}{3}(k+1)^3$

[Standard wording]

$\frac{1}{4}n^4 < \sum_{r=1}^{n} r^3 \leq n^4$

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Solution

[Show that the result is true for $n = 1$]

Now assume that the result is true for $n = k$

so that $\frac{1}{4}k^4 < \sum_{r=1}^{k} r^3 \leq k^4$

Then $\sum_{r=1}^{k+1} r^3 > \frac{1}{4}k^4 + (k+1)^3$

$= \frac{1}{4}(k+1)^4 - \frac{1}{4}(4k^3 + 6k^2 + 4k + 1) + (k+1)^3$

$= \frac{1}{4}(k+1)^4 - \frac{1}{4}(6k^2 + 4k + 1) + 3k^2 + 3k + 1$

$= \frac{1}{4}(k+1)^4 + \frac{3}{2}k^2 + 2k + \frac{3}{4}$

$> \frac{1}{4}(k+1)^4$ (as $k > 0$)

Also, $\sum_{r=1}^{k+1} r^3 \leq k^4 + (k+1)^3$

$= (k+1)^4 - 4k^3 - 6k^2 - 4k - 1 + (k+1)^3$

$= (k+1)^4 - 4k^3 - 6k^2 - 4k - 1 + k^3 + 3k^2 + 3k + 1$

$= (k+1)^4 - 3k^3 - 3k^2 - k$

$< (k+1)^4$ (as $k > 0$)

$\leq (k+1)^4$

[Standard wording]

The sum of the interior angles of a convex n-sided polygon is $180(n-2)$

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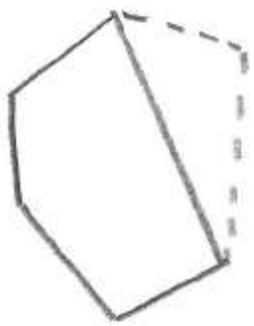
Solution

When $n = 3$ (the smallest possible value), the result is true, as the interior angles of a triangle add up to $180°$.

Now assume that the result is true for $n = k$, so that the total of the interior angles is $180(k-2)$.



The diagram shows the case $k = 5$, but applies more generally.

By adding another triangle, n has increased by 1, and the total of the interior angles has increased by 180.

Thus the total for $k+1$ sides is $180(k-2) + 180 = 180(k-1)$

$= 180([k+1] - 2)$

[Standard wording, but starting at $n = 3$]

If $A = \begin{pmatrix} -1 & -4 \\ 1 & 3 \end{pmatrix}$, then $A^n = \begin{pmatrix} 1-2n & -4n \\ n & 1+2n \end{pmatrix}$

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Solution

[Show that the result is true for $n = 1$]

Now assume that the result is true for $n = k$

so that $A^k = \begin{pmatrix} 1-2k & -4k \\ k & 1+2k \end{pmatrix}$

Target is: $A^{k+1} = \begin{pmatrix} -1-2k & -4k-4 \\ k+1 & 3+2k \end{pmatrix}$

$A^{k+1} = \begin{pmatrix} -1 & -4 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1-2k & -4k \\ k & 1+2k \end{pmatrix}$

$= \begin{pmatrix} -1+2k-4k & 4k-4-8k \\ 1-2k+3k & -4k+3+6k \end{pmatrix}$

$= \begin{pmatrix} -1-2k & -4k-4 \\ k+1 & 3+2k \end{pmatrix}$

[Standard wording]

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\ldots\left(1-\frac{1}{n^2}\right)=\frac{n+1}{2n}\text{ for }n\geq 2$$

$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\ldots\left(1-\frac{1}{n^2}\right)=\frac{n+1}{2n}$ for $n \geq 2$

Solution

[Show that the result is true for $n = 2$]

Now assume that the result is true for $n = k$

so that $\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\ldots\left(1-\frac{1}{k^2}\right)=\frac{k+1}{2k}$

Target: $\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\ldots\left(1-\frac{1}{(k+1)^2}\right)=\frac{k+2}{2k+2}$

LHS $=\left(\frac{k+1}{2k}\right)\left(1-\frac{1}{(k+1)^2}\right)=\left(\frac{k+1}{2k}\right)\left(\frac{(k+1)^2-1}{(k+1)^2}\right)$

$=\left(\frac{1}{2k}\right)\left(\frac{k^2+2k}{k+1}\right)=\frac{k(k+2)}{k(2k+2)}=\frac{k+2}{2k+2}$

[Standard wording, but starting at $n = 2$]