Induction – Exercises: Divisibility (18 pages ; 30/7/25)

$7^{2n-1} + 3^{2n}$ is divisible by 8

$7^{2n-1} + 3^{2n}$ is divisible by 8

Solution

[Show that the result is true for $n = 1$]

Now assume that the result is true for $n = k$

Approach 1

so that $7^{2k-1} + 3^{2k} = 8M$, where $M \in \mathbb{Z}^+$

To show that the result is then true for $n = k + 1$:

$7^{2(k+1)-1} + 3^{2(k+1)} = 7^{2k+1} + 3^{2k+2}$

$= 49(7^{2k-1}) + 3^{2k+2}$

$= 49(8M - 3^{2k}) + 3^{2k+2}$

$= 8(49M) + 3^{2k}(-49 + 9)$

$= 8(49M - 5(3^{2k}))$

(the multiple is positive, as $7^{2(k+1)-1} + 3^{2(k+1)}$ is positive)

[Standard wording]

Approach 2

Note: This approach is sometimes suggested with $\lambda = 1$, but there is no guarantee that it will work for this value of λ, as can be seen in subsequent examples.

Let $f(k) = 7^{2k-1} + 3^{2k}$

Then $f(k+1) - \lambda f(k) = 7^{2k+1} + 3^{2k+2} - \lambda(7^{2k-1} + 3^{2k})$

$= 7^{2k-1}(49 - \lambda) + 3^{2k}(9 - \lambda)$

Let $\lambda = 1$, so that $f(k+1) = f(k) + 48(7^{2k-1}) + 8(3^{2k})$

As $f(k)$ is assumed to be a multiple of 8, and the other terms on the RHS are also a multiple of 8, it follows that $f(k+1)$ is a multiple of 8. [Standard wording]

$2^{n+2} + 3^{2n+1}$ is divisible by 7

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Solution

[Show that the result is true for $n = 1$]

Now assume that the result is true for $n = k$

Approach 1

so that $2^{k+2} + 3^{2k+1} = 7M$, where $M \in \mathbb{Z}^+$

To show that the result is then true for $n = k + 1$:

$2^{(k+1)+2} + 3^{2(k+1)+1} = 2^{k+3} + 3^{2k+3}$

$= 2(2^{k+2}) + 3^{2k+3} = 2(7M - 3^{2k+1}) + 3^{2k+3}$

$= 7(2M) + 3^{2k+1}(-2 + 9)$

$= 7(2M + 3^{2k+1})$

[Standard wording]

Approach 2

Let $f(k) = 2^{k+2} + 3^{2k+1}$

Then $f(k+1) - \lambda f(k) = 2^{k+3} + 3^{2k+3} - \lambda(2^{k+2} + 3^{2k+1})$

$= 2^{k+2}(2 - \lambda) + 3^{2k+1}(9 - \lambda)$

Let $\lambda = 2$, so that $f(k+1) = 2f(k) + 7(3^{2k+1})$

As both terms on the RHS are multiples of 7, it follows that $f(k+1)$ is a multiple of 7.

[Note that we cannot set $\lambda = 1$ in this example.]

[Standard wording]

$5^n + 12n - 1$ is divisible by 16

$5^n + 12n - 1$ is divisible by 16

Solution

[Show that the result is true for $n = 1$]

Now assume that the result is true for $n = k$

Approach 1

so that $5^k + 12k - 1 = 16M$, where $M \in \mathbb{Z}^+$

To show that the result is then true for $n = k + 1$:

$5^{k+1} + 12(k + 1) - 1 = 5(16M - 12k + 1) + 12k + 11$

$= 16(5M) - 48k + 16$

$= 16(5M - 3k + 1)$

(the multiple is positive, as $5^{k+1} + 12(k + 1) - 1$ is positive)

[Standard wording]

Approach 2

Let $f(k) = 5^k + 12k - 1$

Then $f(k + 1) - \lambda f(k)$

$= 5^{k+1} + 12(k + 1) - 1 - \lambda(5^k + 12k - 1)$

$= 5^k(5 - \lambda) + 12k(1 - \lambda) + 11 + \lambda$

Let $\lambda = 5$, so that $f(k + 1) = 5f(k) - 48k + 16$

As all the terms on the RHS are multiples of 16, it follows that $f(k + 1)$ is a multiple of 16.

[Standard wording]

$2^{n+1} + 9(13^n)$ is divisible by 11

$2^{n+1} + 9(13^n)$ is divisible by 11

Solution

[Show that the result is true for $n = 1$]

Now assume that the result is true for $n = k$

Approach 1

so that $2^{k+1} + 9(13^k) = 11M$, where $M \in \mathbb{Z}^+$

To show that the result is then true for $n = k + 1$:

$2^{k+2} + 9(13^{k+1}) = 2\{11M - 9(13^k)\} + 9(13^{k+1})$

$= 11(2M) + 13^k\{9(13) - 18)\}$

$= 11(2M) + 99(13^k)$

$= 11(2M + 9(13^k))$

[Standard wording]

Approach 2

Let $f(k) = 2^{k+1} + 9(13^k)$

Then $f(k + 1) - \lambda f(k)$

$= 2^{k+2} + 9(13^{k+1}) - \lambda(2^{k+1} + 9(13^k))$

$= 2^{k+1}(2 - \lambda) + 13^k(117 - 9\lambda)$

Let $\lambda = 2$, so that $f(k + 1) = 2f(k) + 99(13^k)$

As both terms on the RHS are multiples of 11, it follows that $f(k + 1)$ is a multiple of 11.

[Standard wording]

$13^n + 6^{n-1}$ is divisible by 7

$13^n + 6^{n-1}$ is divisible by 7

Solution

[Show that the result is true for $n = 1$]

Now assume that the result is true for $n = k$

Approach 1

so that $13^k + 6^{k-1} = 7M$, where $M \in \mathbb{Z}^+$

To show that the result is then true for $n = k + 1$:

$13^{k+1} + 6^k = 13(7M - 6^{k-1}) + 6^k$

$= 7(13M) + 6^{k-1}(6 - 13)$

$= 7(13M - 6^{k-1})$

(the multiple is positive, as $13^{k+1} + 6^k$ is positive)

[Standard wording]

Approach 2

Let $f(k) = 13^k + 6^{k-1}$

Then $f(k + 1) - \lambda f(k)$

$= (13^{k+1} + 6^k) - \lambda(13^k + 6^{k-1})$

$= 13^k(13 - \lambda) + 6^{k-1}(6 - \lambda)$

putting $\lambda = 13$ [or $\lambda = -1$]

$= -7(6^{k-1})$, so that $f(k + 1) = 13f(k) - 7(6^{k-1})$

As both terms on the RHS are multiples of 7, it follows that $f(k + 1)$ is a multiple of 7

(the multiple is positive, as $13^{k+1} + 6^k$ is positive)

[Standard wording]

$5^{2n} + 12^{n-1}$ is divisible by 13

$5^{2n} + 12^{n-1}$ is divisible by 13

Solution

[Show that the result is true for $n = 1$]

Now assume that the result is true for $n = k$

Approach 1

so that $5^{2k} + 12^{k-1} = 13M$, where $M \in \mathbb{Z}^+$

To show that the result is then true for $n = k + 1$:

$5^{2k+2} + 12^k = 25(13M - 12^{k-1}) + 12^k$

$= 13(25M) + 12^{k-1}(12 - 25)$

$= 13(25M - 12^{k-1})$

(the multiple is positive, as $5^{2k+2} + 12^k$ is positive)

[Standard wording]

Approach 2

Let $f(k) = 5^{2k} + 12^{k-1}$

Then $f(k + 1) - \lambda f(k)$

$= 5^{2k+2} + 12^k - \lambda(5^{2k} + 12^{k-1})$

$= 5^{2k}(25 - \lambda) + 12^{k-1}(12 - \lambda)$

putting $\lambda = 25$ [or $\lambda = -1$]

$= -(13)12^{k-1}$, so that $f(k + 1) = 25f(k) - 13(12^{k-1})$

As both terms on the RHS are multiples of 13, it follows that
$f(k + 1)$ is a multiple of 13

(the multiple is positive, as $5^{2k+2} + 12^k$ is positive)

[Standard wording]

$5^{2n+2} - 24n - 25$ is divisible by 576

$5^{2n+2} - 24n - 25$ is divisible by 576

Solution

[Show that the result is true for $n = 1$]

Now assume that the result is true for $n = k$

Approach 1

so that $5^{2k+2} - 24k - 25 = 576M$, where $M \in \mathbb{Z}^+$

To show that the result is then true for $n = k + 1$:

$5^{2k+4} - 24(k + 1) - 25 = 25(576M + 24k + 25) - 24k - 49$

$= 576(25M) + 24k(24) + 576$

$= 576(25M + k + 1)$

[Standard wording]

Approach 2

Let $f(k) = 5^{2k+2} - 24k - 25$

Then $f(k + 1) - \lambda f(k)$

$= 5^{2k+4} - 24(k + 1) - 25 - \lambda(5^{2k+2} - 24k - 25)$

$= 5^{2k+2}(25 - \lambda) + k(-24 + 24\lambda) - 49 + 25\lambda$

putting $\lambda = 25$

$= 24(24)k - 49 + 625 = 576(k + 1)$

so that $f(k + 1) = 25f(k) + 576(k + 1)$

As both terms on the RHS are multiples of 576, it follows that $f(k + 1)$ is a multiple of 576

[Standard wording]

$2^{4n+1} + 3$ is divisible by 5

$2^{4n+1} + 3$ is divisible by 5

Solution

[Show that the result is true for $n = 1$]

Now assume that the result is true for $n = k$

Approach 1

so that $2^{4k+1} + 3 = 5M$, where $M \in \mathbb{Z}^+$

To show that the result is then true for $n = k + 1$:

$2^{4k+5} + 3 = 16(5M - 3) + 3$

$= 5(16M - 9)$

(the multiple is positive, as $2^{4k+5} + 3$ is positive)

[Standard wording]

Approach 2

Let $f(k) = 2^{4k+1} + 3$

Then $f(k + 1) - \lambda f(k)$

$= (2^{4k+5} + 3) - \lambda(2^{4k+1} + 3)$

$= 2^{4k+1}(16 - \lambda) + 3(1 - \lambda)$

putting $\lambda = 16$ [or $\lambda = 1$]

$= -45$

so that $f(k + 1) = 16f(k) - 45$

As both terms on the RHS are multiples of 5, it follows that $f(k + 1)$ is a multiple of 5 (and the multiple is positive, as

$2^{4k+5} + 3$ is positive)

[Standard wording]