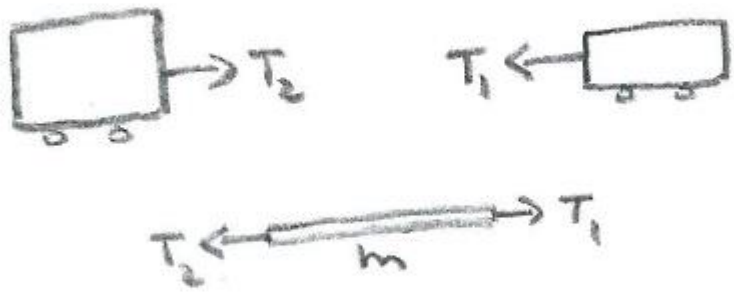
Impulses & connected particles (13 pages; 28/8/18)

(1) (Introduction) Tensions and compressions in rods

Consider a car pulling a caravan, by means of a (rather long) towbar, as in the diagram below.

Assuming that the car is pulling on the towbar with a force T_1, then the towbar will be pulling on the car with the same force, by Newton's 3rd law (ie the car will experience a drag on it from the caravan, via the towbar).

Similarly, the caravan will experience a pull from the towbar (of T_2, say), and the towbar will experience a pull of the same force from the caravan.



Applying Newton's 2nd law to the towbar, assumed to be of mass m, $T_1 - T_2 = ma$, where a is the acceleration of the car, caravan and towbar.

In order to be able to cope with the general case where $a \neq 0$, we have to make the modelling assumption that the towbar is 'light'; ie of negligible mass, so that $T_1 - T_2 = 0$,

and hence $T_1 = T_2$ ($= T$, say).

Diagrams in textbooks often appear as below (with other forces often included as well). It is intended that the T forces are acting on the car and caravan (rather than the towbar). Strictly speaking, there ought to be separate force diagrams for the car and caravan.

In this example, the towbar is being pulled apart, and is described as being under tension. The opposite situation, where the forces on the towbar are both inwards is referred to as a compression (also known as a 'thrust'), and this terminology is probably more obvious, as the towbar is of course being compressed. This will arise when the car is decelerating at a greater rate than the deceleration that the caravan would experience if it wasn't being pulled (ie due to the various resistance forces on it). (Consider Newton's 2nd law for the caravan: $T - R = Ma$: it is possible for T to be positive (ie the towbar is under tension) when $a < 0$, provided R is large enough).

It is customary when answering Mechanics questions involving tensions or compressions, to assume initially that we are dealing with a tension. Then if T turns out to be negative; eg -1000N , we say that there is a compression (or thrust) of 1000N.

(2) Impulses

Where a force F is being applied to an object for a (usually) short period of time t, there is said to be an impulse Ft on the object.

As force is a vector quantity, impulse is as well. Also, Newton's 3rd law effectively applies to impulses, as t will be common to both objects. Thus, if P exerts an impulse J on Q, then Q exerts an equal and opposite impulse J on P. (Incidentally, J is often used to represent an impulse, rather than I, as I is reserved for moment of inertia).

This note concerns situations where two particles are connected by a light rod or inelastic string. As rods can be under either tension or compression (or thrust), we can talk about there being an impulsive tension or thrust (thrust is now used in preference to compression) in the rod if it is subjected to a force for a short period of time.

Inelastic strings can be under tension, but not compression, and so only an impulsive tension is possible.

(3) **Example A**

In the diagram below, particle P (of mass M) is connected to particle Q (of mass m) by a rod. P is given an impulse of J, which acts at an angle θ to PQ, as shown.

The impulse J has a component acting along PQ, and so the rod is under an impulsive thrust (of K, say), and by Newton's 3rd law, the rod exerts an impulse K on P.

By the same reasoning as for the towbar example, the rod also exerts an impulse K on Q.

The impulse-momentum equation can be applied to 'impulse diagrams'.

For P: $\begin{pmatrix} Jcos\theta \\ Jsin\theta \end{pmatrix} + \begin{pmatrix} -K \\ 0 \end{pmatrix} = M \begin{pmatrix} v_x \\ v_y \end{pmatrix}$ (1)

A special feature of these problems is that the component of the velocity of P along the rod (ie v_x), immediately after the impulse, will be the same as that of Q. (See (6) for a discussion of the situation involving an inelastic string.)

For Q: $\begin{pmatrix} K \\ 0 \end{pmatrix} = m \begin{pmatrix} v_x \\ 0 \end{pmatrix}$ (2)

[Note that Q only experiences an impulse along the rod, and so will have no component of velocity in any other direction.]

From (1) & (2),

$Jcos\theta - K = Mv_x$ (3)

$Jsin\theta = Mv_y$ (4)

$K = mv_x$ (5)

Then, for example, to find the direction in which P moves after the impulse, we can write $tan\phi = \frac{v_y}{v_x}$, where ϕ is the angle that the path of P makes with the x-axis.

From (3) & (5), $Jcos\theta - mv_x = Mv_x$, and hence $v_x = \frac{Jcos\theta}{M+m}$

Then, from (4), $tan\phi = \frac{v_y}{v_x} = \frac{\left(\frac{Jsin\theta}{M}\right)}{\left(\frac{Jcos\theta}{M+m}\right)} = \frac{(M+m)tan\theta}{M}$

[This seems reasonable: if m is small relative to M, then $tan\phi \approx tan\theta$, whilst if M is small relative to m, then $tan\phi$ is large; ie $\phi \approx 90°$ (consider the case where Q is a brick wall: P wouldn't have an x component of velocity.]

Alternative approach

If we aren't required to find the impulsive tension (or thrust), then we can often consider the system as a whole (treating the impulsive tensions as internal impulses that cancel out), and apply the impulse-momentum equation (assuming there are no external forces acting, other than those producing the impulses).

For this example, the overall impulse-momentum equation is:

$$\begin{pmatrix} Jcos\theta \\ Jsin\theta \end{pmatrix} = M\begin{pmatrix} v_x \\ v_y \end{pmatrix} + m\begin{pmatrix} v_x \\ 0 \end{pmatrix}$$

(with the same considerations affecting the velocity of Q)

So $Jcos\theta = (M+m)v_x$ and $Jsin\theta = Mv_y$,

giving $\frac{v_y}{v_x} = \frac{\left(\frac{Jsin\theta}{M}\right)}{\left(\frac{Jcos\theta}{M+m}\right)} = \frac{(M+m)tan\theta}{M}$, as before.

[We can in fact then find K from $K = mv_x$]

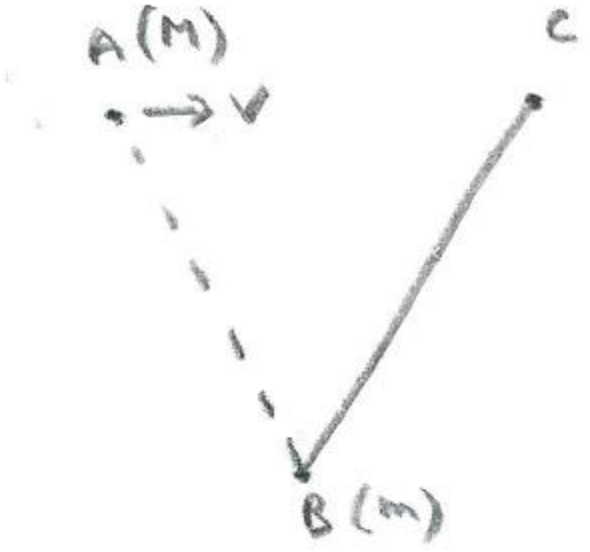
(4) **Example B**

Referring to the diagram below, particles A and B are connected by an inelastic string that is taut initially. A is given an impulse in the direction of C, where ABC is an equilateral triangle, so as to give it a speed V. Let J be the impulsive tension in the string when A reaches C.

(i) Show that $J = \frac{mMV}{2(M+m)}$

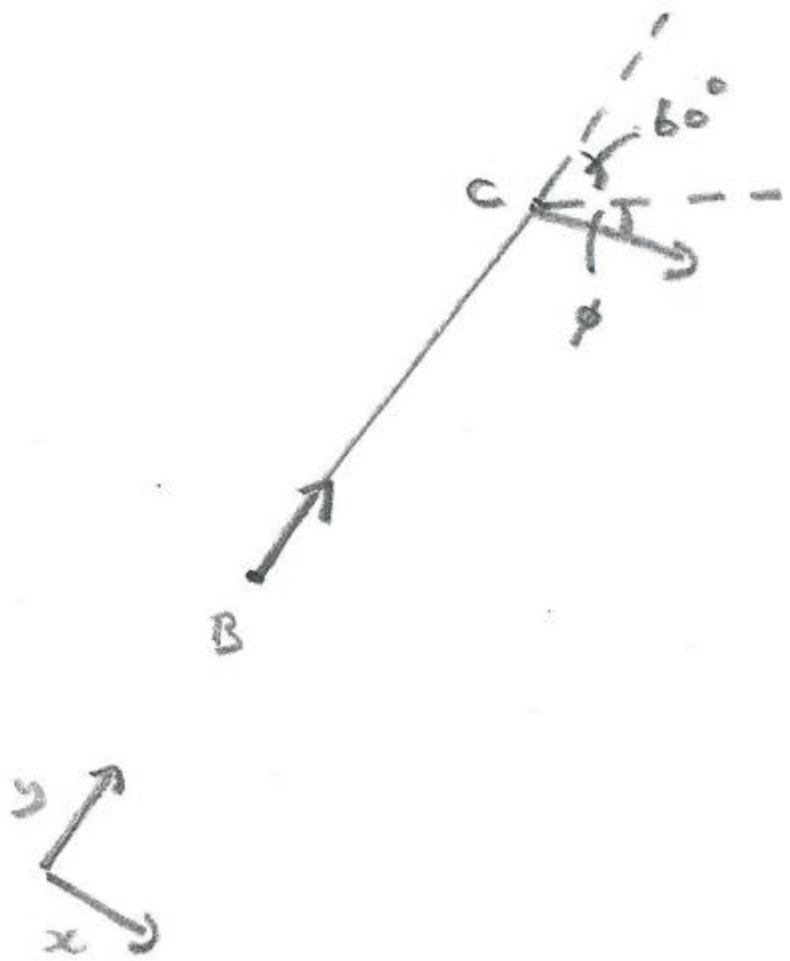
(ii*) Show that, after A reaches C, its direction of motion is at an angle ϕ below the line AC, where $tan\phi = \frac{m\sqrt{3}}{4M+3m}$

(iii) Find an expression for the kinetic energy lost by the system as a result of the impulsive tension created.



Solution

(i) Note that the string is slack until A reaches C, so that B doesn't move until then.



Let the directions x & y be as shown in the diagram above. The velocities of A and B just after C has been reached (ie when the impulsive tension has taken effect) can then be given in component form as follows:

A: $\begin{pmatrix} v_x \\ v_y \end{pmatrix}$, B: $\begin{pmatrix} 0 \\ v_y \end{pmatrix}$

[The x component of B is 0, as there is no impulse on B in that direction; the y component of B is v_y, as A and B are assumed to move with the same speed along the string, when it is taut (ie just as for the rod situation). See (6) for a discussion of this assumption.

Note: Although A is being pulled towards B (when it reaches C), a component of its original momentum is in the positive y direction, which enables the component of A's velocity after reaching C to be in the positive y direction.]

Conservation of momentum for A then gives:

$$\begin{pmatrix} 0 \\ -J \end{pmatrix} = M\begin{pmatrix} v_x \\ v_y \end{pmatrix} - M\begin{pmatrix} Vsin60° \\ Vcos60° \end{pmatrix}$$

and for B: $\begin{pmatrix} 0 \\ J \end{pmatrix} = m\begin{pmatrix} 0 \\ v_y \end{pmatrix}$

This then gives the 3 equations:

$0 = Mv_x - \frac{\sqrt{3}}{2}MV$ or $v_x = \frac{\sqrt{3}}{2}V$ (1)

$-J = Mv_y - \frac{1}{2}MV$ (2)

$J = mv_y$ (3)

Then J can be found by eliminating v_y from (2) & (3):

$-J = M\left(\frac{I}{m}\right) - \frac{1}{2}MV$

$\Rightarrow \frac{1}{2}MV = J(\frac{M}{m} + 1)$

$\Rightarrow J = \frac{MV}{2(\frac{M}{m}+1)} = \frac{mMV}{2(M+m)}$

(ii) Referring to the previous diagram, $\tan(60° + \phi) = \frac{v_x}{v_y}$

From (1), $v_x = \frac{\sqrt{3}}{2}V$ and from (3), $v_y = \frac{J}{m} = \frac{MV}{2(M+m)}$

Hence $\frac{tan60° + tan\phi}{1 - tan60° tan\phi} = \frac{\sqrt{3}(M+m)}{M}$

$\Rightarrow \frac{\sqrt{3}+tan\phi}{1-\sqrt{3}tan\phi} = \sqrt{3}\lambda$, where $\lambda = \frac{M+m}{M}$

$\Rightarrow \sqrt{3} + tan\phi = \sqrt{3}\lambda - 3\lambda tan\phi$

$\Rightarrow tan\phi(1 + 3\lambda) = \sqrt{3}(\lambda - 1)$

$\Rightarrow tan\phi = \frac{\sqrt{3}(\lambda-1)}{1+3\lambda} = \frac{\sqrt{3}(\frac{m}{M})}{1+\frac{3(M+m)}{M}} = \frac{\sqrt{3}m}{M+3M+3m} = \frac{m\sqrt{3}}{4M+3m}$

Alternative approach for (b)

As before, we can bypass the internal impulses, by considering conservation of momentum for the overall system, noting that AC makes an angle of 30° with the x-direction (by dropping a perpendicular from A to BC):

$$M\begin{pmatrix} Vcos30° \\ Vsin30° \end{pmatrix} = M\begin{pmatrix} v_x \\ v_y \end{pmatrix} + m\begin{pmatrix} 0 \\ v_y \end{pmatrix}$$

Then $v_x = \frac{\sqrt{3}}{2}V$ and $v_y = \frac{MV}{2(M+m)}$, as before.

(iii) Loss of kinetic energy $= \frac{1}{2}MV^2 - \frac{1}{2}M\left({v_x}^2 + {v_y}^2\right) - \frac{1}{2}m{v_y}^2$

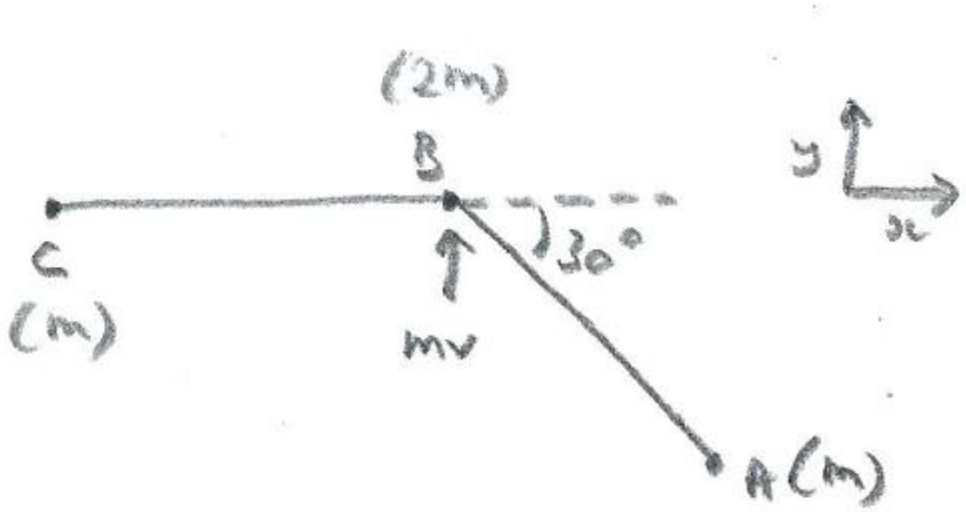
$= \frac{1}{2}MV^2 - \frac{1}{2}M\left(\frac{\sqrt{3}}{2}V\right)^2 - \frac{1}{2}(M+m)(\frac{MV}{2(M+m)})^2$

$= \frac{1}{8}MV^2 - \frac{1}{8}\frac{M^2V^2}{M+m} = \frac{1}{8}MV^2\left(1 - \frac{M}{M+m}\right) = \frac{1}{8}MV^2(\frac{m}{M+m})$

[Note that this is zero when $m = 0.$]

(5) **Example C**

Referring to the diagram below, the particles A, B & C, of masses m, $2m$ & m respectively, lie at rest on a smooth table, and are connected, as shown, by taut inelastic strings. An impulse of mv is applied to B, in a direction perpendicular to CB and away from A. Find the initial velocities of the particles, immediately after the impulse.



Solution

Particle A will move along AB (with speed v_a, say), and particle C will move along CB (with speed v_c, say). Let $\binom{v_x}{v_y}$ be the velocity of particle B, where the x and y directions are as shown in the diagram. This velocity can be resolved along and perpendicular to AB, and the component in the direction AB must equal v_a.

This gives: $-v_x cos30° + v_y cos60° = v_a$ (1)

Also, the component in the direction CB (v_x) is assumed to equal v_c. (2)

In addition, the impulse-momentum equation can be applied to the whole system:

$$\begin{pmatrix} 0 \\ mv \end{pmatrix} = m\begin{pmatrix} -v_a cos30° \\ v_a sin30° \end{pmatrix} + 2m\begin{pmatrix} v_x \\ v_y \end{pmatrix} + m\begin{pmatrix} v_c \\ 0 \end{pmatrix} \quad (3)$$

(1), (2) give $-v_c\frac{\sqrt{3}}{2} + v_y(\frac{1}{2}) = v_a$, so that $v_y = 2v_a + v_c\sqrt{3}$ (4)

Then (2), (3) & (4) give:

$0 = -v_a\frac{\sqrt{3}}{2} + 2v_c + v_c$ and $v = \frac{1}{2}v_a + 2(2v_a + v_c\sqrt{3})$,

so that $6v_c = v_a\sqrt{3}$

and $2v = 9v_a + 4v_c\sqrt{3} = 9v_a + 4\left(\frac{v_a\sqrt{3}}{6}\right)\sqrt{3} = 11v_a$

Thus $v_a = \frac{2v}{11}$ and $v_c = \frac{2v}{11}\frac{\sqrt{3}}{6} = \frac{v}{11\sqrt{3}}$

Also $\begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} \frac{v}{11\sqrt{3}} \\ 2v_a + v_c\sqrt{3} \end{pmatrix} = \begin{pmatrix} \frac{v}{11\sqrt{3}} \\ \frac{v}{11}(4+1) \end{pmatrix} = \frac{v}{11}\begin{pmatrix} \frac{1}{\sqrt{3}} \\ 5 \end{pmatrix}$

from which the magnitude and direction of B's velocity can be determined, if necessary.

(6) Theoretical considerations

(i) Conservation of energy

As was seen in Example B, conservation of energy can't generally be used. Even if a system is not subject to any external forces (and the system does no work), it can still lose energy (for example, consider the collision of two balls on a smooth surface, when energy will be lost unless $e = 1$).

However, it seems that it is possible to use conservation of energy for problems involving connected particles when there are no impulsive tensions or thrusts in the rod or string (and no work is done to or by the system)(see Wragg, "Modern Mechanics", p173, Example 9).

(ii) In Examples B and C (involving inelastic strings), we assumed that the components of the velocities of the particles along the string (immediately after the impulse) were the same. This has to be true in the case of a rod, but isn't quite as obvious for an inelastic string.

First of all, note that the initial tension might disappear at some subsequent point, but we are only concerned with the immediate velocities of the particles.

Suppose that (as in Example B), the two particles are A and B, with B being the one at the free end of the string, and let the components of the velocities of A and B along the sting (immediately after the impulse) be v_a and v_b.

As the string is inelastic, it is definitely true that $v_b \geq v_a$.

Textbooks always assert that $v_b = v_a$, but this is questioned in an internet article by C.T.O'Sullivan ("Impulsive_tensions_in_strings-a_century_of_misconception"), which compares the motion of B with that of a ball involved in a direct collision with another ball, where the coefficient of restitution is not 1. In that case, energy is lost, and the ball's speed is not the same as it would be if $e = 1$.

Mr. O'Sullivan makes the case for a similar treatment for impulsive tensions:

> Finally, it is clear that the concept of a 'coefficient of restitution' is just as applicable to impulsive tensions in inextensible strings as it is to collisions between rigid bodies.

...

> impulse. Interestingly, none of the authors quoted above suggest that such a common velocity can be determined in the case of the direct impact of rigid spheres where the same argument would apply.

The idea (I assume) is that some of the kinetic energy that has been lost might be available (for a suitable string) to increase v_b, so that $v_b > v_a$. However, this alternative model probably wouldn't score any marks in an A Level exam.