Impulse & Momentum Exercises (Solutions)

(8 pages; 19/3/20)

(1**) Two particles of the same mass are travelling directly towards each other, on a smooth surface. Particle A has a speed which is k times that of particle B (where $k > 0$).

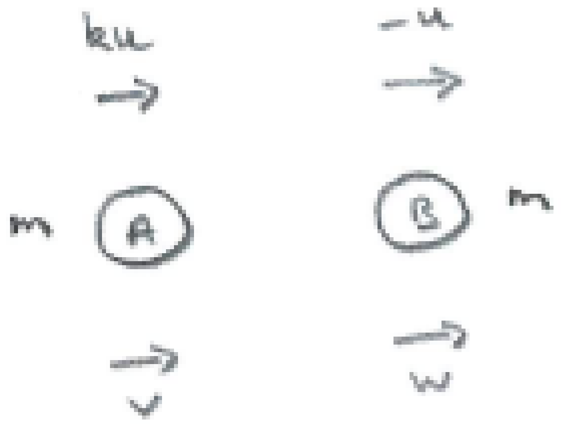
(i) Find the condition on k that must apply in order for A to change direction on impact.

(ii) Describe the motion of the particles after they have collided, in the case where $e = 0$.

(iii) Describe the motion of the particles after they have collided, in the case where $e = 1$.

(iv) In the case where $e = \frac{1}{3}$, describe the motion of the particles after they have collided, for the various possible values of k.

Solution



Conservation of momentum $\Rightarrow m(ku - u) = m(v + w)$, where m is the mass of each particle, so that $(k-1)u = v + w$ (1)

By Newton's Law of Restitution, $w - v = e(ku - (-u))$,

so that $eu(k+1) = w - v$ (2)

Adding (1) & (2), $u(k-1+ek+e)=2w$ (3)

Subtracting (2) from (1), $u(k-1-ek-e)=2v$ (4)

(i) From (4), $v<0 \Rightarrow k-1-ek-e<0$ (as $u>0$)

$\Rightarrow k(1-e)<e+1$

$\Rightarrow k<\frac{1+e}{1-e}$, provided $e\neq 1$ (as $1-e>0$)

[If e is close enough to 1, A will reverse its direction for any value of k (the bigger k is, the closer e has to be to 1).]

(ii) When $e=0$, (4) & (3) $\Rightarrow v=w=\frac{(k-1)u}{2}$

Thus the particles coalesce, and travel in the original direction of the particle with the bigger speed.

(iii) When $e=1$, (4) & (3) $\Rightarrow v=-u$ and $w=ku$

Thus both A and B have reversed their directions, and exchanged speeds.

(iv) When $e=\frac{1}{3}$, (4) & (3) $\Rightarrow v=\frac{u}{2}\left(\frac{2}{3}k-\frac{4}{3}\right)$ and $w=\frac{u}{2}\left(\frac{4}{3}k-\frac{2}{3}\right)$

$v<0$ when $\frac{2}{3}k-\frac{4}{3}<0$; ie $k<2$

$w<0$ when $\frac{4}{3}k-\frac{2}{3}<0$; ie $k<\frac{1}{2}$

So, when $k<\frac{1}{2}$, both A and B move off to the left (in the diagram).

When $k=\frac{1}{2}$, A moves off to the left, and B comes to rest.

When $\frac{1}{2} < k < 2$, A moves off to the left and B moves off to the right.

When $k = 2$, A comes to rest, and B moves off to the right.

When $k > 2$, both A and B move off to the right.

(2**) For two balls colliding directly on a smooth surface, show that kinetic energy is conserved when $e = 1$.

Solution

Let the two balls have masses m_A & m_B, initial speeds u_A & u_B and final speeds v_A & v_B (where the speeds are from left to right, and $u_A > 0$, with $u_A > u_B$).

Then, by conservation of momentum,

$m_A u_A + m_B u_B = m_A v_A + m_B v_B \quad (1)$

and, by Newton's law of impact, $\frac{v_B - v_A}{u_A - u_B} = e = 1 \quad (2)$

Result to prove: $\frac{1}{2} m_A ({v_A}^2 - {u_A}^2) + \frac{1}{2} m_B ({v_B}^2 - {u_B}^2) = 0 \quad (3)$

From (1), $m_B (v_B - u_B) = m_A (u_A - v_A)$,

and from (2), $(v_B + u_B) = (u_A + v_A)$.

Then, substituting into (3),

$LHS = \frac{1}{2} m_A (v_A - u_A)(v_A + u_A) + \frac{1}{2} m_B (v_B - u_B)(v_B + u_B)$

$= \frac{1}{2} m_A (v_A - u_A)(v_A + u_A) + \frac{1}{2} m_A (u_A - v_A)(u_A + v_A) = 0$,

as required.

(3***) A spaceship has a geostationary orbit about the earth (ie it stays above the same point on the earth's surface). An astronaut walks from one end of the spaceship to the other. Describe what happens, relative to the earth's surface.

Solution

Let the spaceship, excluding the astronaut, have mass M, and let the astronaut have mass m. Suppose that the astronaut is walking with velocity w relative to the spaceship, and that the spaceship (including the astronaut) travels at velocity v relative to the earth's surface, once the astronaut has started walking.

By conservation of momentum,

$Mv + m(v + w) = 0 \quad \Rightarrow v(M + m) = -mw$

and so $v = -\frac{mw}{(M+m)}$

ie the spaceship moves in the opposite direction to the motion of the astronaut relative to the spaceship.

Consider the motion of the centre of mass of the spaceship and astronaut.

Its velocity relative the the earth's surface is the weighted average of the velocities of the spaceship (excluding the astronaut) and the astronaut:

$\left(\frac{M}{M+m}\right) v + \left(\frac{m}{M+m}\right)(v + w)$

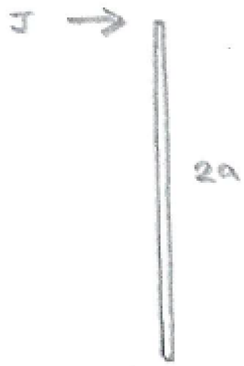
$= \left(\frac{1}{M+m}\right)(Mv + mv + mw)$

$= v + \frac{mw}{M+m} = 0$

As there is no external force on the spaceship and astronaut, we would expect there to be no net motion of the centre of mass of the spaceship and astronaut.

(4***) Impulse on Rod

An impulse J is applied to one end of a thin, uniform rod of length $2a$ and mass m, as shown below. Describe the resulting motion.



Solution

By conservation of linear momentum, if v is the velocity of the centre of mass of the rod after the impulse, then:

$J = mv$ (1)

And by conservation of angular momentum, if ω is the angular velocity about the centre of mass after the impulse, then

$aJ = I\omega$ (2),

where I, the moment of inertia of the rod about an axis through the centre of mass, perpendicular to the rod $= \frac{1}{3}ma^2$

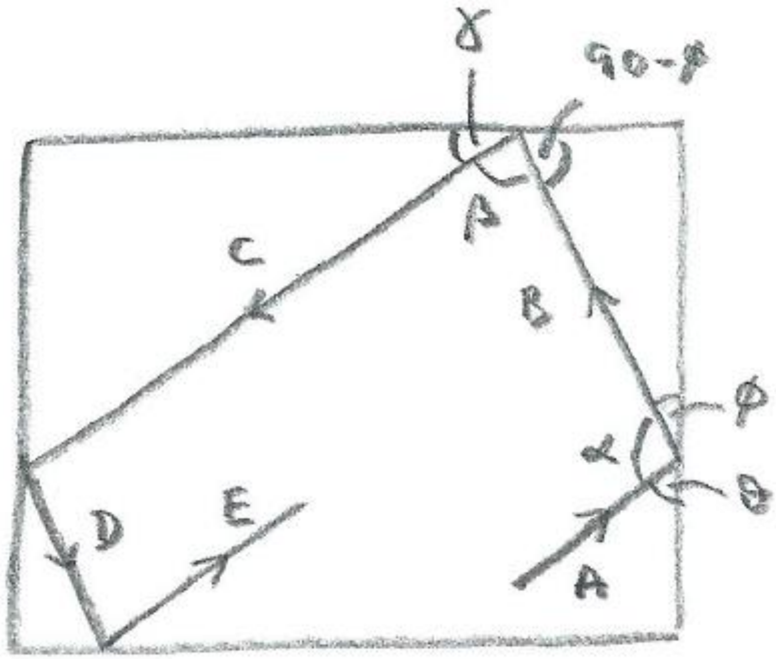
So, the motion of the rod after the impulse is a combination of a velocity of $v = \frac{J}{m}$ in the direction of the impulse, together with a rotation about the centre of mass, with angular velocity

$\omega = \frac{aJ}{\left(\frac{1}{3}ma^2\right)}$

$= \frac{3J}{ma}$

(5***) A snooker ball is hit towards a cushion, with speed v, in such a way that it hits each of the four sides of the table. The coefficient of restitution between the ball and the cushions is e. Investigate the speed and direction of the ball.

Solution



(a)(i) Referring to the diagram, when the ball is at A (travelling towards the 1st cushion), its velocity vector is $\binom{vsin\theta}{vcos\theta}$, and the gradient of its path is $cot\theta$.

(ii) When the ball is at B (travelling towards the 2nd cushion), its velocity vector is $\binom{-evsin\theta}{vcos\theta}$, and the gradient of its path is $-\frac{1}{e}cot\theta$.

(iii) To find the relation between θ and ϕ:

(a) See note on Oblique impacts, which shows that $tan\phi = etan\theta$

(b) This can be verified by considering the slope at B:

$$\tan\phi = \frac{evsin\theta}{vcos\theta} = etan\theta$$

(c) A more complicated approach is:

$$cos\phi = \frac{\binom{-evsin\theta}{vcos\theta}.\binom{0}{1}}{v\sqrt{e^2sin^2\theta+cos^2\theta}(1)} = \frac{vcos\theta}{v\sqrt{e^2sin^2\theta+cos^2\theta}} = \frac{cos\theta}{\sqrt{e^2sin^2\theta+cos^2\theta}}$$

$$\Rightarrow cos^2\phi = \frac{cos^2\theta}{e^2sin^2\theta+cos^2\theta} = \frac{1}{e^2tan^2\theta+1}$$

$$\Rightarrow e^2tan^2\theta + 1 = sec^2\phi = tan^2\phi + 1$$

$$\Rightarrow e^2tan^2\theta = tan^2\phi$$

$\Rightarrow tan\phi = etan\theta$ (as $e > 0$ and $\theta, \phi < 90°$)

(iv) When the ball is at C (travelling towards the 3rd cushion), its velocity vector is $\binom{-evsin\theta}{-evcos\theta}$, and the gradient of its path is $cot\theta$.

So the path at C is parallel to that at A; ie it has turned through 180°.

It follows that $\alpha + \beta = 180°$ (from the properties of parallel lines).

(v) The speed of the ball at C is $\sqrt{(-evsin\theta)^2 + (-evcos\theta)^2}$

$= ev$

(vi) To find an expression for γ:

$\gamma + \beta + (90 - \phi) = 180$

$\Rightarrow \gamma = 90 - \beta + \phi = 90 - (180 - \alpha) + \phi$

$= \alpha + \phi - 90$

$= (180 - \theta - \phi) + \phi - 90$

$= 90 - \theta$

(vii) When the ball is at D (travelling towards the 4th cushion), its velocity vector is $\begin{pmatrix} e^2vsin\theta \\ -evcos\theta \end{pmatrix}$, and the gradient of its path is $-\frac{1}{e}cot\theta$. So the path at D is parallel to that at B.

(viii) When the ball is at E (travelling away from the 4th cushion), its velocity vector is $\begin{pmatrix} e^2vsin\theta \\ e^2vcos\theta \end{pmatrix}$, and the gradient of its path is $cot\theta$. So the path at E is parallel to that at A.

(ix) The speed of the ball at E is e^2v .