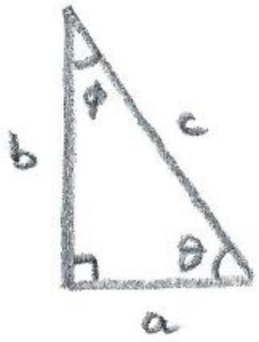
Important Ideas - Trigonometry (7 pages; 24/10/20)

(1) Relation between sin and cos [STEP/MAT]



Referring to the diagram,

$sin\theta = \frac{b}{c} = cos\phi = cos(90° - \theta)$

and $cos\theta = \frac{a}{c} = sin\phi = sin(90° - \theta)$

(The 'co' in cosine stands for 'complementary', because θ and $90° - \theta$ are described as complementary angles.)

(2) Key Results [STEP/MAT]

(A) Compound Angle formulae

$sin(\theta + \phi) = sin\theta cos\phi + cos\theta sin\phi$

$cos(\theta + \phi) = cos\theta cos\phi - sin\theta sin\phi$

(B) $sin(\theta \pm 360°) = sin\theta$; $cos(\theta \pm 360°) = cos\theta$

$cos(-\theta) = cos\theta$; $sin(-\theta) = -sin\theta$

$sin(180° - \theta) = sin\theta$; $cos(180° - \theta) = -cos\theta$

$sin\theta = \cos(90° - \theta)$; $cos\theta = sin(90° - \theta)$

(C) Translations

$sin(\theta + 90°)$ is $sin\theta$ translated 90° to the left, which is $cos\theta$

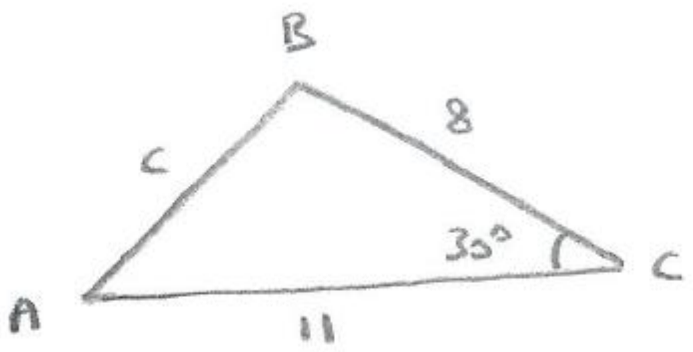
$sin(\theta - 90°)$ is $sin\theta$ translated 90° to the right, which is $-cos\theta$

$cos(\theta + 90°)$ is $cos\theta$ translated 90° to the left, which is $-sin\theta$

$cos(\theta - 90°)$ is $cos\theta$ translated 90° to the right, which is $sin\theta$

(3) As $sin\theta = sin(180° - \theta)$, we have to be careful when using the Sine rule to determine angles in a triangle that are close to 90°. Instead, either find small angles first, or use the Cosine rule instead.

Example



$c^2 = 11^2 + 8^2 - 2(11)(8)cos30°$, giving $c = 5.70785$

Now $\frac{sinB}{11} = \frac{sin30°}{5.70785} \Rightarrow sinB = 0.96359 \Rightarrow B = 74.5°$ or $105.5°$

But $\frac{sinA}{8} = \frac{sin30°}{5.70785} \Rightarrow sinA = 0.70079$

$\Rightarrow A = 44.5°$ (not $180 - 44.5$)

$\Rightarrow B = 180 - 30 - 44.5 = 105.5°$

(4) To solve eg $\sin(2x - 60°) = 0.5$; $0 \leq x \leq 360°$:

Let $u = 2x - 60°$ and note that $-60° \leq u \leq 660°$

Having found the solutions for u (such that $-60° \leq u \leq 660°$), the solutions for x are obtained from $x = \frac{1}{2}(u + 60)$.

(5) Starting with $cos^2\theta + sin^2\theta = 1$ (A) and

$cos^2\theta - sin^2\theta = cos2\theta$ (B),

$\frac{1}{2}[(A) + (B)] \Rightarrow cos^2\theta = \frac{1}{2}(1 + cos2\theta)$

and $\frac{1}{2}[(A) - (B)] \Rightarrow sin^2\theta = \frac{1}{2}(1 - cos2\theta)$

(6)(a) In order for $y = arcsinx$ (or $sin^{-1}x$) to be a function, the range of the inverse of $y = sinx$ is restricted to $[-\frac{\pi}{2}, \frac{\pi}{2}]$

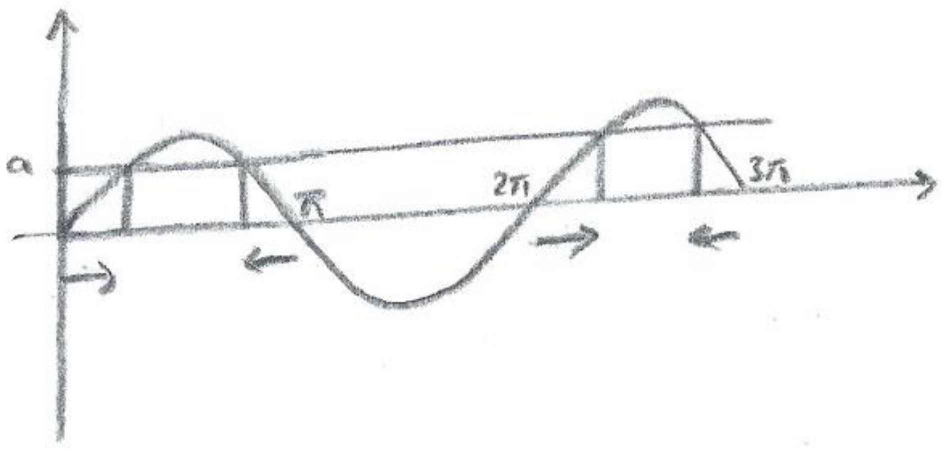
(To avoid vertical duplication for $y = arcsinx$, we ensure that there is no horizontal duplication for $y = sinx$.)

Then $sinx = a \Rightarrow$

$x = arcsin(a) + n(2\pi)$ or $\pi - arcsin(a) + n(2\pi)$ for $n \in \mathbb{Z}$

Alternatively, $x = n\pi + (-1)^n arcsin(a)$

[For even multiples of π, we go forward along the curve, and for odd multiples we go back - see the diagram below.]



(b) In order for $y = arctanx$ (or $tan^{-1}x$) to be a function, the range of the inverse of $y = tanx$ is also restricted to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Then $tanx = a \Rightarrow x = arctan(a) + n\pi$ for $n \in \mathbb{Z}$

(c) In order for $y = arccosx$ (or $cos^{-1}x$) to be a function, the range of the inverse of $y = cosx$ is restricted to $[0, \frac{\pi}{2}]$

(avoiding horizontal duplication for $y = cosx$)

Then $cosx = a \Rightarrow$

$x = arccos(a) + n(2\pi)$ or $2\pi - arccos(a) + n(2\pi)$ for $n \in \mathbb{Z}$

[The 2nd option can also be written as $-arccos(a) + n'(2\pi)$]

Alternatively, $x = 2n\pi \pm arccos(a)$

(7) Why radians are usually preferred to degrees

(i) The key point is that $sin\theta \approx \theta$ for small θ measured in radians, but, if ϕ is the equivalent angle measured in degrees, then

$$sin\phi = sin\theta \approx \theta = \left(\frac{\pi}{180}\right)\phi$$

So $\lim\limits_{\theta \to 0} \frac{sin\theta}{\theta} = 1$ when θ is measured in radians, but $\lim\limits_{\phi \to 0} \frac{sin\phi}{\phi} = \frac{\pi}{180}$ when ϕ is measured in degrees.

(ii) With the same notation, consider the graph of $sin\phi$ and compare it with the graph of $sin\theta$.
$sin\phi$ increases from 0 to 1 as ϕ increases from 0° to 90°, whereas $sin\theta$ increases from 0 to 1 as θ increases from 0 to $\frac{\pi}{2}$.

Thus the graph of $sin\phi$ is more stretched out than that of $sin\theta$, with a much smaller gradient (except when the gradient is zero).

In particular, at the Origin, $y = sin\theta$ tends to $y = \theta$ only when θ is measured in radians.

(iii) $\frac{d}{d\theta} sin\theta = \lim\limits_{h \to 0} \frac{\sin(\theta + h) - sin\theta}{h} = \lim\limits_{h \to 0} \frac{sin\theta\cos(h) + cos\theta\sin(h) - sin\theta}{h}$

One version of l'H$\hat{o}$pital's rule states that,

if $\lim\limits_{x \to a} f(x) = \lim\limits_{x \to a} g(x) = 0$, and if $\lim\limits_{x \to a} \frac{f(x)}{g(x)}$ exists,

then $\lim\limits_{x \to a} \frac{f(x)}{g(x)} = \lim\limits_{x \to a} \frac{f'(x)}{g'(x)}$

(the other version applies when $\lim\limits_{x \to a} f(x) = \pm\infty$ & $\lim\limits_{x \to a} g(x) = \pm\infty$)

Hence, $\lim\limits_{h \to 0} \frac{sin\theta cos(h) + cos\theta sin(h) - sin\theta}{h}$

$= \lim\limits_{h \to 0} \frac{sin\theta(cos(h) - 1)}{h} + \lim\limits_{h \to 0} \frac{cos\theta sin(h)}{h}$ (A)

As $sin\theta(\cos(h) - 1) \to 0$ as $h \to 0$,

and $\frac{d}{dh}\left(sin\theta(\cos(h) - 1)\right) = sin\theta(-sin(h))$, whilst $\frac{d}{dh}(cos\theta sin(h)) = cos\theta cos(h)$,

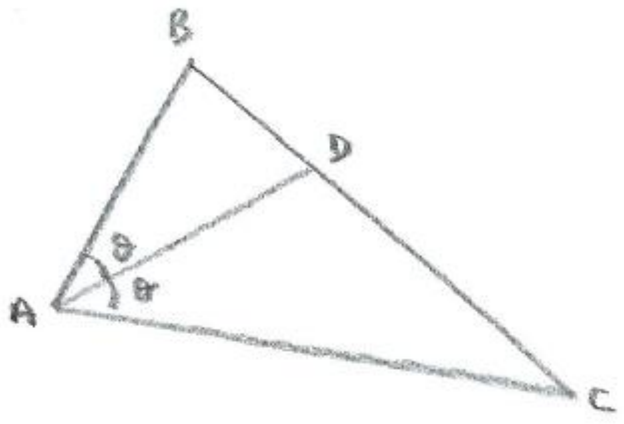
(A) equals $\lim_{h \to 0} \frac{sin\theta(-sin(h))}{1} + \lim_{h \to 0} \frac{cos\theta cos(h)}{1}$

$= 0 + cos\theta$, giving $\frac{d}{d\theta} sin\theta = cos\theta$, provided that θ is in radians.

(8) Angle Bisector Theorem [MAT/STEP]

Referring to the diagram below, the Angle Bisector theorem says that

$\frac{BD}{DC} = \frac{AB}{AC}$



Proof

Method 1

By the Sine rule for triangle ABD, $\frac{BD}{sin\theta} = \frac{AB}{sinADB}$ (1)

and, for triangle ADC, $\frac{DC}{sin\theta} = \frac{AC}{sinADC} = \frac{AC}{sinADB}$ (2)

Then (1) $\Rightarrow \frac{sin\theta}{sinADB} = \frac{BD}{AB}$ and (2) $\Rightarrow \frac{sin\theta}{sinADB} = \frac{DC}{AC}$

so that $\frac{BD}{AB} = \frac{DC}{AC}$

and hence $\frac{BD}{DC} = \frac{AB}{AC}$

Method 2

Area of triangle ABD ÷ Area of triangle ADC = $\frac{\frac{1}{2}AB.ADsin\theta}{\frac{1}{2}AC.ADsin\theta} = \frac{AB}{AC}$

Also,

Area of triangle ABD ÷ Area of triangle ADC = $\frac{\frac{1}{2}BD.ADsinBDA}{\frac{1}{2}AD.DCsinADC} = \frac{BD}{DC}$,

as $\angle BDA = 180 - \angle ADC$, so that $sinBDA = sinADC$

Hence, $\frac{AB}{AC} = \frac{BD}{DC}$