Series - Important Ideas (3 pages ; 11/3/21)

(1) $\sum_{r=1}^{n} r = 1 + 2 + 3 + \cdots + n = \frac{1}{2}\, n(n+1)$

[Informal proof: The average size of the terms being added is $\frac{1}{2}(1+n)$, and there are n terms.]

(2) Arithmetic sequences and series

[Note: 'series' ⇒ the terms of the sequence are added.]

Inductive/iterative/recurrence formula

$a_{r+1} = a_r + d\ ;\ a_1 = a$

Deductive/direct formula

$a_r = a + (r-1)d = dr + (a-d) = dr + a_0$

[compare with the straight line $y = mx + c$, with $x = r, y = a_r$, $m = d$ & $c = a_0 = a_1 - d$; noting that a_0 is the hypothetical '0th' term of the sequence.]

(3) Method of Differences

Example: To find $\sum_{r=1}^{n} \frac{1}{(r+1)(r+2)(r+3)}$

Step 1: Writing $\frac{1}{(r+1)(r+2)(r+3)} = \frac{A}{r+1} + \frac{B}{r+2} + \frac{C}{r+3}$ (method of Partial Fractions),

RHS $= \frac{A(r+2)(r+3)+B(r+1)(r+3)+C(r+1)(r+2)}{(r+1)(r+2)(r+3)}$, so that

$$1 = A(r+2)(r+3) + B(r+1)(r+3) + C(r+1)(r+2) \quad (*)$$

Setting $r = -2$, $1 = -B; B = -1$

Setting $r = -3$, $1 = 2C$; $C = \frac{1}{2}$

Setting $r = -1$, $1 = 2A$; $A = \frac{1}{2}$

[As a check, equating the coefficients of r^2 (as (*) has to hold for all r), $0 = A + B + C$]

Thus $\frac{1}{(r+1)(r+2)(r+3)} = \frac{1}{2(r+1)} - \frac{1}{r+2} + \frac{1}{2(r+3)}$

Step 2: $\sum_{r=1}^{n} \frac{1}{(r+1)(r+2)(r+3)} = \frac{1}{2}\sum_{r=1}^{n}\{\frac{1}{(r+1)} - \frac{2}{(r+2)} + \frac{1}{(r+3)}\} = \frac{1}{2}S$,

where $S = (\frac{1}{2} + \frac{1}{3} + \frac{\mathbf{1}}{\mathbf{4}} + \frac{\mathbf{1}}{\mathbf{5}} + \cdots + \frac{\mathbf{1}}{\boldsymbol{n}} + \frac{\mathbf{1}}{\boldsymbol{n}+\mathbf{1}})$

$-2(\frac{1}{3} + \frac{\mathbf{1}}{\mathbf{4}} + \frac{\mathbf{1}}{\mathbf{5}} + \cdots + \frac{\mathbf{1}}{\boldsymbol{n}} + \frac{\mathbf{1}}{\boldsymbol{n}+\mathbf{1}} + \frac{1}{n+2})$

$+(\frac{\mathbf{1}}{\mathbf{4}} + \frac{\mathbf{1}}{\mathbf{5}} + \cdots + \frac{\mathbf{1}}{\boldsymbol{n}} + \frac{\mathbf{1}}{\boldsymbol{n}+\mathbf{1}} + \frac{1}{n+2} + \frac{1}{n+3})$

The bold terms cancel, so that

$\sum_{r=1}^{n} \frac{1}{(r+1)(r+2)(r+3)} = \frac{1}{2}\left\{\frac{5}{6} - \frac{2}{3} - \frac{1}{n+2} + \frac{1}{n+3}\right\}$

$= \frac{1}{12} - \frac{1}{2(n+2)} + \frac{1}{2(n+3)}$

Notes

(i) Avoid writing $S = \left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{2}{4} + \frac{1}{5}\right) + \cdots$, as it complicates the cancelling.]

(ii) The method of differences relies on there being a suitable form for the partial fractions!

(4) Power series

(i) Maclaurin: $f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \cdots$

[As a first approximation, $f'(0) \approx \frac{f(x)-f(0)}{x}$ (for small x)]

(ii) Taylor I: $f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots$

[As a first approximation, $f'(a) \approx \frac{f(x)-f(a)}{x-a}$ (for x close to a)]

(iii) Taylor II: $f(x+a) = f(a) + xf'(a) + \frac{x^2}{2!}f''(a) + \cdots$

[As a first approximation, $f'(a) \approx \frac{f(x+a)-f(a)}{x}$ (for small x);

$a = 0$ gives the Maclaurin series]

(5) $\sum_{r=1}^{\infty} ra^r = a\frac{d}{da}\sum_{r=1}^{\infty} a^r = a\frac{d}{da}(\frac{a}{1-a})$ (when $|a| < 1$)

$= a\,.\frac{(1-a)-a(-1)}{(1-a)^2} = \frac{a}{(1-a)^2}$