Important Ideas - Proof (3 pages; 22/10/20)

(1) Necessary and sufficient conditions

To prove that $A \Leftrightarrow B$, we can either prove that $A \Rightarrow B$ and

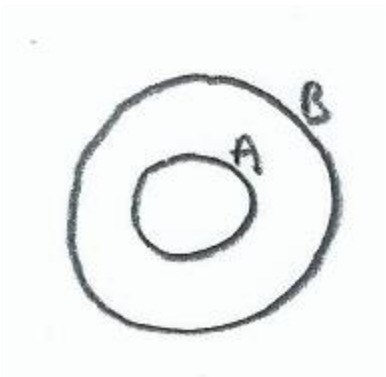
$B \Rightarrow A$, or instead that $A \Rightarrow B$ and $A' \Rightarrow B'$

To show that $A' \Rightarrow B'$ implies that $B \Rightarrow A$:

If B is true, suppose that A is not true. Then, as $A' \Rightarrow B'$, there is a contradiction, as B is true. So A must be true, and hence $B \Rightarrow A$.]

Alternative approach

$A \Rightarrow B$ can be interpreted as "A is a subset of B". So, referring to the diagram, $A \Leftrightarrow B$ means that the gap between A and B is an empty set, and this is the effect of $A' \Rightarrow B'$.



Example 1

Let A be the statement: $X + Y < XY + 1$,

let B be the statement: $X > 1$ & $Y > 1$,

and let C be the statement: $X < 1$ & $Y < 1$

$A \equiv X - XY < 1 - Y$

$\equiv X(1 - Y) - (1 - Y) < 0$

$\equiv (X - 1)(1 - Y) < 0$

$\equiv (X-1)(Y-1) > 0$

So the following are true:

$B \Rightarrow A$ (B is a sufficient condition for A; A is a necessary condition for B)

$A \nRightarrow B$ (A is not a sufficient condition for B; B is not a necessary condition for A)

$A \Leftrightarrow B\ or\ C$

Example 2

Let A be the statement: The transformation represented by the 2×2 matrix $\underline{A}$ has a line of invariant points that does not pass through the Origin;

let B be the statement: $\underline{A} = \underline{I}$ (for 2×2 matrices)

It can be shown that $A \Leftrightarrow B$

[$B \Rightarrow A$ follows from the fact that every point is invariant for the identity transformation; for a proof that $A \Rightarrow B$, see sol'n to STEP 2019, P3, Q3(i)]

Example 3 [STEP]

(from STEP 2018, P2, Q7)

Show that, if c & d are non-zero real numbers, then distinct real numbers p & q can be found, such that $c = pq$ & $d = pq(p+q)$, provided that $d^2 > 4c^3$.

Solution

Suppose that $c = pq$ and $d = pq(p+q)$ where $p \neq q$ (A)

Then $d = c(p+q)$ and so $d = c(p + \frac{c}{p})$

and $dp = cp^2 + c^2$, so that $cp^2 - dp + c^2 = 0$ (B)

This has distinct real sol'ns when $d^2 - 4c^3 > 0$ (C)

and, by symmetry, the sol'ns of (B) will be p & q.

Thus, provided that $d^2 > 4c^3$, there will be distinct solutions p & q of $cx^2 - dx + c^2 = 0$, with $pq = \frac{c^2}{c} = c$

and $p + q = \frac{-(-d)}{c}$, so that $d = pq(p + q)$;

ie c & d can be expressed in terms of p & q, as required.

[Note that we must be careful to show that (C) ⇒(A), rather than the other way round; although, in order to derive the proof, we needed to proceed from (A) to (C).]