Important Ideas - Polynomials (2 pages; 4/1/21)

(1) Factorisation of polynomials

(1)(i) $x^2 - y^2 = (x - y)(x + y)$

(ii) $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$

$x^3 + y^3 = (x + y)(x^2 - xy + y^2)$

[Let $f(x) = x^3 - y^3$. Then $f(y) = 0$, and so $x - y$ is a factor of $x^3 - y^3$, by the Factor Theorem.]

(iii) $x^n - y^n = (x - y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1})$

or $(x + y)(x^{n-1} - x^{n-2}y + \cdots + xy^{n-2} - y^{n-1})$, if n is even

$x^n + y^n = (x + y)(x^{n-1} - x^{n-2}y + \cdots - xy^{n-2} + y^{n-1})$ if n is odd

Summary

Factor:	$x - y$	$x + y$
$x^n - y^n$; odd n	Yes (A)	**No (B)**
$x^n - y^n$; even n	Yes (C)	Yes (D)
$x^n + y^n$; odd n	No (P)	**Yes (Q)**
$x^n + y^n$; even n	No (R)	No (S)

[The 'exceptions' are highlighted. As an aid to memory, the familiar factorisations $x^2 - y^2 = (x - y)(x + y)$ and

$x^3 - y^3 = (x - y)(x^2 + xy + y^2)$ are examples of (C), (D) & (A), suggesting that $x^3 - y^3 = (x + y)$ ... (ie (B)) is the one that isn't possible for $x^n - y^n$. This then prompts us to recall that

$x^3 + y^3 = (x + y)$ ... (ie (Q)) is the one that **is** possible for $x^n + y^n$.]

(2) Integer roots of Polynomials

Let $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$

where $n \geq 2$ and the a_i are integers, with $a_0 \neq 0$.

Then it can be shown that any rational root of the equation $f(x) = 0$ will be an integer. [Based on STEP 2011, P3, Q2]

Proof

Suppose that there is a rational root $\frac{p}{q}$, where p & q are integers with no common factor greater than 1 and $q > 0$.

Then $\left(\frac{p}{q}\right)^n + a_{n-1}\left(\frac{p}{q}\right)^{n-1} + \cdots + a_2\left(\frac{p}{q}\right)^2 + a_1\left(\frac{p}{q}\right) + a_0 = 0$

and, multiplying by q^{n-1}:

$\frac{p^n}{q} + a_{n-1}p^{n-1} + a_{n-2}p^{n-2}q + \cdots + a_1pq^{n-2} + a_1q^{n-1} = 0$

Then, as all the terms from $a_{n-1}p^{n-1}$ onwards are integers, it follows that $\frac{p^n}{q}$ is also an integer, and hence $q = 1$ (as p & q have no common factor greater than 1), and the root is an integer.