

## Important Ideas: Integration Methods (14 pages; 28/10/20)

[The constant of integration is generally omitted in these notes.]

(See also Pure: "Integration Exercises" (Parts 1-4).)

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### (A) Substitutions

(1) Substitutions may be considered to be of the following types (with possible overlap):

(a) A linear substitution (ie of the form  $u = ax + b$ ) which will simplify the integrand (the expression being integrated), without introducing any unwanted complications from the change of variable (as  $du = adx$ ).

(b) A 'speculative' substitution, which has a good chance of simplifying the integrand, even after the complications from the change of variable.

(c) Integrals of the form  $\int f(x)h(g(x)) dx$ , where

$\int f(x)dx = g(x)$ , where it will be seen that the substitution  $u = g(x)$  will work, provided that  $h(u)$  can be integrated.

(d) Substitutions in definite integrals

The limits may provide a clue as to a suitable substitution.

## (2) Linear substitutions

**Example:**  $I = \int x(1+x)^{\frac{1}{2}} dx$

Note that  $(1+x)^{\frac{1}{2}}$  can only be expanded as an infinite Binomial series. However, let  $u = 1+x$ , giving  $I = \int (u-1)u^{\frac{1}{2}} du$ , and the integrand can now be expanded.

## (3) 'Speculative' substitutions

**Example:**  $I = \int \sqrt{1-x^2} dx$

Let  $x = \sin\theta$ , so that  $dx = \cos\theta d\theta$

Then  $I = \int \cos\theta \cos\theta d\theta = \frac{1}{2} \int 1 + \cos 2\theta d\theta$

$$= \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta = \frac{1}{2}\theta + \frac{1}{2}\sin\theta \cos\theta$$

$$= \frac{1}{2}\arcsin x + \frac{1}{2}x\sqrt{1-x^2}$$

(as can be confirmed by differentiation)

**Example:**  $I = \int \frac{1}{x\sqrt{1-x^2}} dx$

The following is a potentially useful substitution.

Let  $u = 1/x$  so that  $du = -1/x^2 dx$  and  $dx = -x^2 du$ ,

so that  $I = -\int \frac{ux^2}{\sqrt{1-\frac{1}{u^2}}} du = -\int \frac{u^2 x^2}{\sqrt{u^2-1}} du$

$$= -\int \frac{1}{\sqrt{u^2-1}} du = -\operatorname{arcosh} u = -\operatorname{arcosh}(1/x)$$

**Example:**  $I = \int \sqrt{x^2 + 2x + 5} dx = \int \sqrt{(x+1)^2 + 2^2} dx$

Let  $x+1 = 2\sinh y$ , so that  $dx = 2\cosh y dy$

$$\begin{aligned}
\text{and } I &= 2 \int \sqrt{\sinh^2 y + 1} (2 \cosh y) dy \\
&= 4 \int \cosh^2 y dy = 2 \int 1 + \cosh(2y) dy \\
&= 2y + \sinh(2y) \\
&= 2 \operatorname{arsinh} \left( \frac{x+1}{2} \right) + 2 \sinh y \cosh y \\
&= 2 \operatorname{arsinh} \left( \frac{x+1}{2} \right) + (x+1) \sqrt{\sinh^2 y + 1} \\
&= 2 \operatorname{arsinh} \left( \frac{x+1}{2} \right) + (x+1) \sqrt{\left( \frac{x+1}{2} \right)^2 + 1}
\end{aligned}$$

**Example:**  $I = \int \sqrt{x^2 + 8x + 7} dx = \int \sqrt{(x+4)^2 - 3^2} dx$

Let  $x + 4 = 3 \cosh y$ , so that  $dx = 3 \sinh y dy$

$$\begin{aligned}
\text{and } I &= 3 \int \sqrt{\cosh^2 y - 1} (3 \sinh y) dy \\
&= 9 \int \sinh^2 y dy \\
&= \frac{9}{2} \int (\cosh(2y) - 1) dy
\end{aligned}$$

(and then similarly to the previous example)

(4) For integrals of the form  $I = \int f(x)h(g(x)) dx$ , where  $\int f(x)dx = ag(x)$ , make the substitution  $u = g(x)$  if  $\int h(u)du$  can be determined.

Then  $du = g'(x) dx = \frac{1}{a} f(x) dx$ , so that  $I = a \int h(u) du$

### Notes

- (i)  $f(x)$  has to be in the numerator of the integrand
- (ii) To find  $\int f(x)dx = g(x)$ , it might be the case that  $g(x)$  appears in a previous part of a question. Differentiate  $g(x)$  to see if this is the case. [See STEP 2016, P2, Q7(iv)]

**Example:**  $I = \int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx$

Integrating  $\sin x$  to give  $-\cos x$  reveals that the substitution

$u = \cos x$  will work:

$du = -\sin x \, dx$ , so that

$$I = -\int \frac{1}{u} \, du = -\ln u = -\ln(\cos x) = \ln(\sec x)$$

**Example:**  $I = \int \sin x \cos^2 x \, dx$

Noting that  $\int \sin x \, dx = -\cos x$ , let  $u = \cos x$ ,

so that  $du = -\sin x \, dx$ , and  $I = -\int u^2 \, du$

**Example:**  $I = \int \frac{\sin x}{\cos^2 x} \, dx$

The substitution  $u = \cos x$  works here as well,

giving  $I = -\int u^{-2} \, du$

**Example:**  $\int \frac{e^x}{e^{2x}+1} \, dx$

Here  $\int e^x \, dx = e^x$ , and we can integrate  $\int \frac{1}{u^2+1} \, du$ , so let  $u = e^x$ .

**Example:**  $I = \int \frac{1}{x \ln x} \, dx$

Noting that  $\int \frac{1}{x} \, dx = \ln x$ , let  $u = \ln x$ , so that  $du = \frac{1}{x} \, dx$ ,

and  $I = \int \frac{1}{u} \, du = \ln(\ln u)$

## Alternative substitutions

$\sec\theta$  can often be used instead of  $\cosh x$ , and  $\tan\theta$  instead of  $\sinh x$ .

### (5) Substitutions in definite integrals

Look for a substitution that reverses the limits (and then take advantage of the fact that  $\int_a^b f(x)dx = -\int_b^a f(x)dx$ ).

$$(i) \int_0^\infty f(x)dx : \text{When } u = \frac{1}{x}, \int_0^\infty \rightarrow \int_\infty^0$$

$$(ii) \int_0^a f(x)dx : \text{When } u = a - x, \int_0^a \rightarrow \int_a^0$$

**Example** (from STEP 2015, P3, Q1)

$$I = \int_0^\infty f\left(x + \frac{1}{x}\right) dx, \quad J = \int_0^\infty \frac{1}{x^2} f\left(x + \frac{1}{x}\right) dx$$

$$\text{Let } u = \frac{1}{x}, \text{ so that } du = -\frac{1}{x^2} dx$$

$$\text{Then } J = \int_\infty^0 f\left(\frac{1}{u} + u\right) (-du) = \int_0^\infty f\left(u + \frac{1}{u}\right) du = I$$

$$[\text{Note that } x + \frac{1}{x} \rightarrow \frac{1}{u} + u]$$

### (B) Rearrangements

$$\text{Example: } \int \frac{1+x}{x-1} dx = \int \frac{x-1}{x-1} dx + \int \frac{2}{x-1} dx \text{ etc}$$

$$\begin{aligned} \text{Example: } \int \frac{1}{1+\cos x} dx &= \int \frac{1-\cos x}{1-\cos^2 x} dx = \int \frac{1-\cos x}{\sin^2 x} dx \\ &= \int \operatorname{cosec}^2 x dx - \int \frac{\cos x}{\sin^2 x} dx \text{ (and these can both be determined)} \end{aligned}$$

**Example:**  $\int \sqrt{1 + \sin 2x} \, dx$

$$1 + \sin 2x = 1 + \cos\left(\frac{\pi}{2} - 2x\right) = 1 + \cos 2\left(\frac{\pi}{4} - x\right)$$

$$= 2\cos^2\left(\frac{\pi}{4} - x\right)$$

**Example:**  $\int \tan^2 \theta \, d\theta = \int \sec^2 \theta - 1 \, d\theta = \tan \theta - \theta$

**Example:**  $I = \int \sec^4 \theta \, d\theta = \int \sec^2 \theta (1 + \tan^2 \theta) \, d\theta$

[Spotting that  $\int \sec^2 \theta \, d\theta = \tan \theta$ ] Let  $u = \tan \theta$ ,  
so that  $du = \sec^2 \theta \, d\theta$ , and  $I = \int 1 + u^2 \, du$

**Example**  $\int \operatorname{sech} x \tanh x \, dx = \int \frac{\sinh x}{\cosh^2 x} \, dx$

and then let  $u = \cosh x$

In the case of hyperbolic functions, there is always the option of invoking the definition in terms of the exponential function.

**Example:**  $\int \operatorname{sech} x \, dx = \int \frac{2}{e^x + e^{-x}} \, dx = 2 \int \frac{e^x}{e^{2x} + 1} \, dx$

Then let  $u = e^x$ , to give  $2 \int \frac{1}{u^2 + 1} \, du = 2 \arctan(e^x)$

**Example:**  $I = \int \frac{1}{x^4 + 1} \, dx = \int \frac{x^{-2}}{x^2 + x^{-2}} \, dx$

[As a speculative substitution] Let  $x = e^y$ , so that  $dx = e^y \, dy$  and  $x^{-1} dx = dy$

Then  $I = \int \frac{e^{-y}}{e^{2y} + e^{-2y}} \, dy = \frac{1}{2} \int \frac{e^{-y}}{\frac{1}{2}(e^{2y} + e^{-2y})} \, dy = \frac{1}{2} \int \frac{\cosh y - \sinh y}{\cosh(2y)} \, dy$

$$= \frac{1}{2} \int \frac{\cosh y}{2\sinh^2 y + 1} dy - \frac{1}{2} \int \frac{\sinh y}{2\cosh^2 y - 1} dy$$

(then let  $u = \sinh y$  &  $u = \cosh y$ , respectively).

### Example:

$$\int \sin(mx) \cos(nx) dx = \frac{1}{2} \int \sin(m+n)x + \sin(m-n)x dx$$

### (C) Integration by Parts

(1) This is an obvious option whenever the integrand is a product of two expressions (or can be rearranged into this form).

However, substitution will also be a possibility, and may well be preferable - especially as there are potential drawbacks with Parts:

(i) It may not be immediately clear which of the two expressions in the product needs to be integrated.

(ii) Applying Parts may not result in an improvement.

(iii) Parts may not work, or we may end up where we started (though there are steps that can be taken to avoid this - see below).

(iv) The process may be time-consuming if Parts has to be applied twice.

(2) The Parts formulae is derived from the product rule for differentiation:

$$\frac{d}{dx}(uv) = \frac{du}{dx}v + u\frac{dv}{dx}, \text{ where } u \text{ \& } v \text{ are functions of } x$$

Integrating both sides then gives

$$uv = \int \frac{du}{dx}v dx + \int u\frac{dv}{dx} dx$$

$$\text{Then } \int \frac{du}{dx} v dx = uv - \int u \frac{dv}{dx} dx$$

However, there is no need to write out all these symbols: note that one of the expressions in the product ( $\frac{du}{dx}$ ) is being integrated (to  $u$ ) and that the integrated expression appears in both the terms on the RHS. The expression that is to be differentiated ( $v$ ) stays the same for the 1st term on the RHS, and is differentiated (to  $\frac{dv}{dx}$ ) only in the 2nd term. In order not to get confused, the two expressions in the original integral could be labelled with an I and a D (indicating that they are to be integrated and differentiated, respectively).

### (3) Standard situation

**Example**  $I = \int x \sin x dx$

In order to obtain a simpler integral on the RHS, we generally want to reduce the power of a term such as  $x^n$  (but see below for an exception). So  $x^n$  should generally be differentiated.

$\sin kx, \cos kx$  &  $e^{kx}$  can be integrated or differentiated.

Here  $I = x(-\cos x) - \int (1)(-\cos x) dx$  etc

**Example**  $I = \int x(3x + 1)^3 dx$

Integrating  $(3x + 1)^3$  and differentiating  $x$ ,

$$I = x \left(\frac{1}{4}\right) (3x + 1)^4 \left(\frac{1}{3}\right) - \int (1) \left(\frac{1}{4}\right) (3x + 1)^4 \left(\frac{1}{3}\right) dx$$

$$= \frac{1}{12} x(3x + 1)^4 - \frac{1}{12} \int (3x + 1)^4 dx$$

$$= \frac{1}{12} x(3x + 1)^4 - \frac{1}{12} \left(\frac{1}{5}\right) (3x + 1)^5 \left(\frac{1}{3}\right)$$

$$= \frac{(3x+1)^4}{180} \{15x - (3x + 1)\} = \frac{(3x+1)^4(12x-1)}{180}$$



**Example:**  $I = \int \sin x \cos x \, dx$

Integrating  $\cos x$  and differentiating  $\sin x$  (to avoid unnecessary minus signs):

$$I = \int \sin x \cos x \, dx = \sin x \cdot \sin x - \int \cos x \cdot \sin x \, dx = \sin^2 x - I$$

$$\text{Hence } I = \frac{1}{2} \sin^2 x$$

**Example:**  $\int \frac{\ln x}{x} \, dx$

Integrating  $\frac{1}{x}$ :

$$I = \int \frac{\ln x}{x} \, dx = \ln x \cdot \ln x - \int \ln x \cdot \frac{1}{x} \, dx = (\ln x)^2 - I$$

$$\text{Hence } I = \frac{1}{2} (\ln x)^2$$

**Example:**  $\int \ln x \, dx$

write as  $\int 1 \cdot \ln x \, dx$

Integrating 1,

$$I = x \ln x - \int x(1/x) \, dx = x \ln x - x$$

(4) When applying Parts twice, the function resulting from integrating one of the components always has to be integrated again (to avoid going round in circles).

**Example:**  $I = \int \sin x \cdot e^x \, dx$

Integrating  $e^x$ , for example,

$$I = \sin x e^x - \int \cos x e^x \, dx$$

Then integrating  $e^x$  again, to apply Parts for a 2nd time:

$$I = \sin x e^x - \{ \cos x e^x - \int (-\sin x) e^x \, dx \}$$

$$= (\sin x - \cos x)e^x - I$$

$$\text{so that } I = \frac{1}{2}(\sin x - \cos x)e^x$$

[If we had chosen to differentiate  $e^x$ , this would have given

$$I = \sin x e^x - \{\sin x e^x - \int \sin x e^x dx\} = I]$$

(5) Definite integral

$$\text{Example: } I = \int_1^e \left(\frac{\ln x}{x}\right)^2 dx$$

Writing the integrand as  $\frac{1}{x^2}(\ln x)^2$  and differentiating  $(\ln x)^2$ :

$$\begin{aligned} I &= \left[-\frac{1}{x}(\ln x)^2\right]_1^e - \int_1^e \left(-\frac{1}{x}\right) 2\ln x \left(\frac{1}{x}\right) dx \\ &= -\left(\frac{1}{e} - 0\right) + 2 \int_1^e \frac{\ln x}{x^2} dx \end{aligned}$$

Then differentiating  $\ln x$ , to apply Parts again:

$$\begin{aligned} I &= -\frac{1}{e} + 2 \left[-\frac{1}{x} \ln x\right]_1^e - 2 \int_1^e \left(-\frac{1}{x}\right) \left(\frac{1}{x}\right) dx \\ &= -\frac{1}{e} - 2 \left(\frac{1}{e} - 0\right) + 2 \int_1^e \frac{1}{x^2} dx \\ &= -\frac{3}{e} + 2 \left[-\frac{1}{x}\right]_1^e = -\frac{3}{e} - 2 \left(\frac{1}{e} - 1\right) = 2 - \frac{5}{e} \end{aligned}$$

(6) Parts doesn't always work:

$$\text{Example: } \int \frac{1}{x \ln x} dx$$

Differentiating  $\frac{1}{\ln x}$ :

$$\begin{aligned} I &= \ln x \cdot \frac{1}{\ln x} - \int \ln x \cdot (-1)(\ln x)^{-2} \left(\frac{1}{x}\right) dx \\ &= 1 + I \quad ?! \end{aligned}$$

The apparent contradiction is due to the constant of integration.

Thus Parts can't be used to find this integral, but the substitution  $u = \ln x$  works, as already seen.

### (D) Reduction formulae

Integration by Parts can sometimes enable a recurrence relation to be set up.

**Example:**  $I_n = \int_0^1 x^n e^{-x} dx$

Integrating  $e^{-x}$  and differentiating  $x^n$  gives:

$$I_n = [ -e^{-x} x^n ]_0^1 - \int_0^1 -nx^{n-1} e^{-x} dx$$

$$= -e^{-1} + 0 + nI_{n-1}$$

Thus  $I_n = nI_{n-1} - e^{-1}$

Then, since  $I_0 = \int_0^1 e^{-x} dx = [ -e^{-x} ]_0^1 = -e^{-1} + 1 = 1 - e^{-1}$ ,

$$I_1 = (1 - e^{-1}) - e^{-1} = 1 - 2e^{-1},$$

$$I_2 = 2(1 - 2e^{-1}) - e^{-1} = 2 - 5e^{-1} \text{ etc}$$

**Example:**  $I_n = \int_0^\pi \cos^n x dx$

Integrating by Parts (writing as  $\cos x \cdot \cos^{n-1} x$  and differentiating  $\cos^{n-1} x$ )

leads to  $I_n = \frac{n-1}{n} I_{n-2}$

Hence  $\int_0^\pi \cos^4 x dx = \frac{3}{4} I_2 = \frac{3}{4} \cdot \frac{1}{2} \int_0^\pi 1 dx = \frac{3\pi}{8}$

**Example:**  $I_n = \int_0^1 \frac{x^n}{\sqrt{1-x}} dx$

$$I_n = [ x^n \cdot \frac{(1-x)^{1/2}}{-1/2} ]_0^1 - \int_0^1 nx^{n-1} \frac{(1-x)^{1/2}}{-1/2} dx$$

$$= 0 + 2n \int_0^1 x^{n-1} \cdot \frac{(1-x)}{(1-x)^{1/2}} dx$$

(forcing the integrand into the form of  $I_n$ )

$$= 2n(I_{n-1} - I_n)$$

$$\Rightarrow (1 + 2n)I_n = 2nI_{n-1}$$

$$\Rightarrow I_n = \frac{2nI_{n-1}}{2n+1}$$

## Notes

(i) Reduction formulae can also be derived for indefinite integrals.

(ii) Sometimes the integrand can be rearranged to give the reduction formula, without performing integration by Parts

$$\text{eg } \int \tan^n x dx = \int \tan^{n-2} x (\sec^2 x - 1) dx$$

$$= \frac{1}{n-1} \tan^{n-1} x - \int \tan^{n-2} x dx \quad (\text{as } \sec^2 x \text{ is the derivative of } \tan x)$$

(iii) Reduction formulae are often associated with a proof by induction.

(iv) If  $I_n = \dots I_{n-2}$ , the result will depend on whether  $n$  is odd or even.

## (E) Substitution $t = \tan\left(\frac{x}{2}\right)$

The substitution  $t = \tan\left(\frac{x}{2}\right)$  is usually a method of last resort: it can convert an integrand involving trig. functions to one involving polynomial expressions.

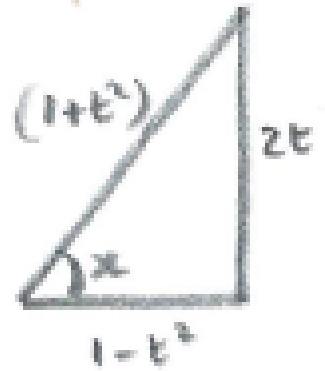
$$t = \tan\left(\frac{x}{2}\right) \Rightarrow \tan x = \frac{2t}{1-t^2}$$

Referring to the right-angled triangle shown,

$$\text{the hypotenuse} = \sqrt{(1-t^2)^2 + 4t^2}$$

$$= \sqrt{1 + 2t^2 + t^4} = 1+t^2 \quad (\text{conveniently})$$

$$\frac{dt}{dx} = \sec^2\left(\frac{x}{2}\right) \cdot \frac{1}{2}, \text{ so that } \frac{dx}{dt} = \frac{2}{\sec^2\left(\frac{x}{2}\right)} = \frac{2}{1+t^2}$$



$$\text{Example: } \int \sec x \, dx = \int \frac{1+t^2}{1-t^2} \cdot \frac{2}{1+t^2} \, dt = 2 \int \frac{1}{1-t^2} \, dt$$

$$= \int \frac{1}{1-t} + \frac{1}{1+t} \, dt = -\ln|1-t| + \ln|1+t| = \ln\left|\frac{1+t}{1-t}\right| = \ln\left|\frac{1+2t+t^2}{1-t^2}\right|$$

$$= \ln\left|\frac{1+t^2}{1-t^2} + \frac{2t}{1-t^2}\right| = \ln|\sec x + \tan x|$$

## (F) Miscellaneous

$$(1) \int_{-a}^a f(-x) \, dx = \int_{-a}^a f(x) \, dx$$

### Proof

Let  $u = -x$ , so that  $du = -dx$ , and

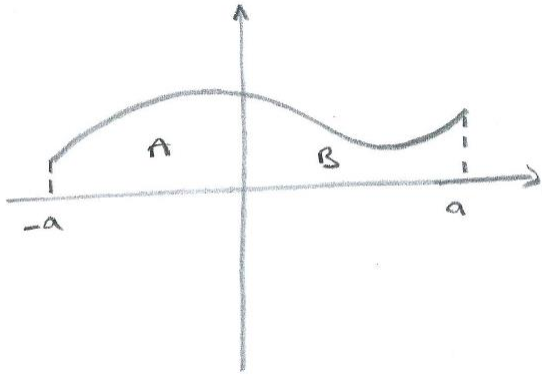
$$\int_{-a}^a f(-x) \, dx = \int_a^{-a} f(u)(-du) = \int_{-a}^a f(u) \, du = \int_{-a}^a f(x) \, dx$$

[Alternatively, considering the integral as an area under a curve, note that  $f(-x)$  is the reflection of  $f(x)$  about the  $y$ -axis, so that

$$\int_{-a}^0 f(-x) \, dx = B = \int_0^a f(x) \, dx \quad (\text{referring to the diagram below})$$

$$\text{and } \int_0^a f(-x) \, dx = A = \int_{-a}^0 f(x) \, dx,$$

$$\text{so that } \int_{-a}^a f(-x) \, dx = B + A = A + B = \int_{-a}^a f(x) \, dx$$



$$(2) \int_0^a f(a-x) dx = \int_0^a f(x) dx$$

**Proof**

Let  $u = a - x$ , so that  $du = -dx$  and

$$\int_0^a f(a-x) dx = \int_a^0 f(u) (-du) = \int_0^a f(u) du = \int_0^a f(x) dx$$

[Note that  $f(a-x)$  is the reflection of  $f(x)$  about  $x = \frac{a}{2}$ .]

(3) To find  $\int \operatorname{cosech}^2 x dx$ , note that  $\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$  and establish that  $\frac{d}{dx}(\coth x) = -\operatorname{cosech}^2 x$