Important Ideas - Induction (2 pages; 22/10/20)

(1) Proof by induction

Example 1: $\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$

Solution

$\sum_{r=1}^{1} r^3 = 1^3 = 1$ and $\frac{1}{4}(1)^2(1+1)^2 = 1$; thus the result is true for $n = 1$

[Be careful to give enough working for both sides.]

Now assume that the result is true for $n = k$, so that

$\sum_{r=1}^{k} r^3 = \frac{1}{4}k^2(k+1)^2$

[A common error is to write "Let $n = k$", or "Assume $n = k$", or "If $n = k$"]

[At this point it is possible to indicate the 'target' for $n = k + 1$; namely that $\sum_{r=1}^{k+1} r^3 = \frac{1}{4}(k+1)^2(k+2)^2$]

Then $\sum_{r=1}^{k+1} r^3 = \frac{1}{4}k^2(k+1)^2 + (k+1)^3$

$= \frac{1}{4}(k+1)^2(k^2 + 4(k+1))$

$= \frac{1}{4}(k+1)^2(k+2)^2$

$= \frac{1}{4}(k+1)^2([k+1]+1)^2$

But this is the result with k replaced by $k + 1$.

[Or, if the target has been mentioned previously, just indicate that the target has been obtained.]

So, if the result is true for $n = k$, then it is true for $n = k + 1$.

[A common error is to write: "So the result is true for $n = k$ and $n = k + 1$"]

As the result is true for $n = 1$, it is therefore true for $n = 2, 3, \ldots$ and, by [the principle of mathematical] induction, for all integer $n \geq 1$. [In some cases, it is appropriate to start at a different value for n, such as 0 or 2. This depends on what values of n the given formula is defined for.]

[Note that no credit is ever given in an exam for the standard wording on its own, or where the algebra is 'fudged'.]

Example 2

(1) $7^n + 4^n + 1$ is divisible by 6

Solution

[Show that the result is true for $n = 1$]

Now assume that the result is true for $n = k$

Approach 1

so that $7^k + 4^k + 1 = 6M$, where $M \in \mathbb{Z}^+$

To show that the result is then true for $n = k + 1$:

$7^{k+1} + 4^{k+1} + 1 = 7(7^k + 4^k + 1) - 3(4^k) - 6$

$= 7(6M) - 6(2)(4^{k-1}) - 6$

$= 6(7M - 2(4^{k-1}) - 1)$, which is a multiple of 6 for $k \geq 1$

(the multiple is positive, as $7^{k+1} + 4^{k+1} + 1$ is positive)

[Standard wording]

Approach 2

Let $f(k) = 7^k + 4^k + 1$

Then $f(k+1) - \lambda f(k) = (7^{k+1} + 4^{k+1} + 1) - \lambda(7^k + 4^k + 1)$

[an appropriate λ will be chosen shortly]

$= 7^k(7-\lambda) + 4^k(4-\lambda) + 1 - \lambda$

Let $\lambda = 7$, so that $f(k+1) - 7f(k) = -3(4^k) - 6$

and $f(k+1) = 7f(k) - 6(2)(4^{k-1}) - 6$

As $f(k)$ is assumed to be a multiple of 6, and the other terms on the RHS are also multiples of 6 (for $k \geq 1$), it follows that $f(k+1)$ is a multiple of 6 (the multiple is positive, as

$7^{k+1} + 4^{k+1} + 1$ is positive).

[Standard wording]

[Textbooks sometimes consider $f(k+1) - f(k)$ (ie with $\lambda = 1$), but this isn't guaranteed to work. (See Induction Exercises for examples of this.)]

(2) 'Weak' and 'strong' induction [STEP]

[$P(k)$ is the proposition that a particular result is true for $n = k$]

'Weak' induction is just the ordinary method; 'strong' induction is where we show that if $P(k-m), P(k-m+1), \dots P(k)$ are correct, then $P(k+1)$ will be correct. We then have to establish that $P(1), P(2), \dots P(m+1)$ are correct. (Weak induction corresponds to $m = 0$.) [In some cases we might start at eg $P(0)$.]

Example of strong induction

g_n is defined recursively as $(n^3 - 3n^2 + 2n)g_{n-3}$ for $n \geq 4$, and $g_1 = 1, g_2 = 2, g_3 = 6$

Show that $g_n = n!$ for $n \geq 1$

Proof

Assume that the result is true for $n = k - 2, k - 1 \; \& \; k$.

Then $g_{k+1} = \left((k+1)^3 - 3(k+1)^2 + 2(k+1)\right)g_{k-2}$

$= (k^3 + 3k^2 + 3k + 1 - 3k^2 - 6k - 3 + 2k + 2)(k-2)!$

$= (k^3 - k)(k-2)!$

$= k(k-1)(k+1)(k-2)!$

$= (k+1)!$

So that the result is true for $n = k + 1$ if it is true for

$n = k - 2, k - 1 \; \& \; k$.

As it is true for $n = 1, 2 \; \& \; 3$, it is therefore true for $n = 4, 5, \ldots$, and hence, by the principle of induction, it is true for all positive integers.