

Important Ideas - Induction (4 pages; 22/10/20)

(1) Proof by induction

Example 1: $\sum_{r=1}^n r^3 = \frac{1}{4}n^2(n+1)^2$

Solution

$\sum_{r=1}^1 r^3 = 1^3 = 1$ and $\frac{1}{4}(1)^2(1+1)^2 = 1$; thus the result is true for $n = 1$

[Be careful to give enough working for both sides.]

Now assume that the result is true for $n = k$, so that

$$\sum_{r=1}^k r^3 = \frac{1}{4}k^2(k+1)^2$$

[A common error is to write "Let $n = k$ ", or "Assume $n = k$ ", or "If $n = k$ "]

[At this point it is possible to indicate the 'target' for $n = k + 1$; namely that $\sum_{r=1}^{k+1} r^3 = \frac{1}{4}(k+1)^2(k+2)^2$]

$$\begin{aligned} \text{Then } \sum_{r=1}^{k+1} r^3 &= \frac{1}{4}k^2(k+1)^2 + (k+1)^3 \\ &= \frac{1}{4}(k+1)^2(k^2 + 4(k+1)) \\ &= \frac{1}{4}(k+1)^2(k+2)^2 \\ &= \frac{1}{4}(k+1)^2([k+1] + 1)^2 \end{aligned}$$

But this is the result with k replaced by $k + 1$.

[Or, if the target has been mentioned previously, just indicate that the target has been obtained.]

So, if the result is true for $n = k$, then it is true for $n = k + 1$.

[A common error is to write: "So the result is true for $n = k$ and $n = k + 1$ "]

As the result is true for $n = 1$, it is therefore true for $n = 2, 3, \dots$ and, by [the principle of mathematical] induction, for all integer $n \geq 1$. [In some cases, it is appropriate to start at a different value for n , such as 0 or 2. This depends on what values of n the given formula is defined for.]

[Note that no credit is ever given in an exam for the standard wording on its own, or where the algebra is 'fudged'.]

Example 2

(1) $7^n + 4^n + 1$ is divisible by 6

Solution

[Show that the result is true for $n = 1$]

Now assume that the result is true for $n = k$

Approach 1

so that $7^k + 4^k + 1 = 6M$, where $M \in \mathbb{Z}^+$

To show that the result is then true for $n = k + 1$:

$$7^{k+1} + 4^{k+1} + 1 = 7(7^k + 4^k + 1) - 3(4^k) - 6$$

$$= 7(6M) - 6(2)(4^{k-1}) - 6$$

$$= 6(7M - 2(4^{k-1}) - 1), \text{ which is a multiple of 6 for } k \geq 1$$

(the multiple is positive, as $7^{k+1} + 4^{k+1} + 1$ is positive)

[Standard wording]

Approach 2

$$\text{Let } f(k) = 7^k + 4^k + 1$$

$$\text{Then } f(k+1) - \lambda f(k) = (7^{k+1} + 4^{k+1} + 1) - \lambda(7^k + 4^k + 1)$$

[an appropriate λ will be chosen shortly]

$$= 7^k(7 - \lambda) + 4^k(4 - \lambda) + 1 - \lambda$$

$$\text{Let } \lambda = 7, \text{ so that } f(k+1) - 7f(k) = -3(4^k) - 6$$

$$\text{and } f(k+1) = 7f(k) - 6(2)(4^{k-1}) - 6$$

As $f(k)$ is assumed to be a multiple of 6, and the other terms on the RHS are also multiples of 6 (for $k \geq 1$), it follows that

$f(k+1)$ is a multiple of 6 (the multiple is positive, as

$7^{k+1} + 4^{k+1} + 1$ is positive).

[Standard wording]

[Textbooks sometimes consider $f(k+1) - f(k)$ (ie with $\lambda = 1$), but this isn't guaranteed to work. (See Induction Exercises for examples of this.)]

(2) 'Weak' and 'strong' induction [STEP]

[$P(k)$ is the proposition that a particular result is true for $n = k$]

'Weak' induction is just the ordinary method; 'strong' induction is where we show that if $P(k-m), P(k-m+1), \dots, P(k)$ are correct, then $P(k+1)$ will be correct. We then have to establish that $P(1), P(2), \dots, P(m+1)$ are correct. (Weak induction corresponds to $m = 0$.) [In some cases we might start at eg $P(0)$.]

Example of strong induction

g_n is defined recursively as $(n^3 - 3n^2 + 2n)g_{n-3}$ for $n \geq 4$, and $g_1 = 1, g_2 = 2, g_3 = 6$

Show that $g_n = n!$ for $n \geq 1$

Proof

Assume that the result is true for $n = k - 2, k - 1$ & k .

$$\begin{aligned} \text{Then } g_{k+1} &= ((k+1)^3 - 3(k+1)^2 + 2(k+1))g_{k-2} \\ &= (k^3 + 3k^2 + 3k + 1 - 3k^2 - 6k - 3 + 2k + 2)(k-2)! \\ &= (k^3 - k)(k-2)! \\ &= k(k-1)(k+1)(k-2)! \\ &= (k+1)! \end{aligned}$$

So that the result is true for $n = k + 1$ if it is true for

$n = k - 2, k - 1$ & k .

As it is true for $n = 1, 2$ & 3 , it is therefore true for $n = 4, 5, \dots$, and hence, by the principle of induction, it is true for all positive integers.