

Hyperbolic Functions: Exercises (4 pages; 20/2/20)

Key to difficulty:

* easier

** moderate

*** harder

(1*) (i) Prove, using exponential functions, that

(a) $\cosh^2 x - \sinh^2 x = 1$

(b) $\sinh 2x = 2 \sinh x \cosh x$

(ii) By differentiating the result from (i)(b), obtain an expression for $\cosh 2x$ in terms of $\cosh^2 x$ and $\sinh^2 x$

(2*) (a) Find the formula connecting $\tanh^2 x$ & $\operatorname{sech}^2 x$?

(b) Find the formula connecting $\operatorname{coth}^2 x$ & $\operatorname{cosech}^2 x$?

(3***) Show that $\operatorname{artanh} x = \frac{1}{2} \ln \left(\frac{1+x}{1-x} \right)$ ($|x| < 1$)

(4**) Differentiation

Find or prove the following:

(i) $\frac{d}{dx} \tanh x$

(ii) $\frac{d}{dx} \operatorname{arcosh} x = \frac{1}{\sqrt{x^2-1}}$

(iii) $\frac{d}{dx} \operatorname{artanh} x = \frac{1}{1-x^2}$

(iv) $\frac{d}{dx} \operatorname{sech} x$

(5*) Simplify $\sinh(\cosh^{-1}2)$

(6**) Solve the equation $5\cosh 2x + 3\sinh x = 6$,
giving your answers in exact logarithmic form

(7***) Show that $\operatorname{arcosh} x = \ln(x + \sqrt{x^2 - 1})$

(8*) If $x = \sinh u$, write $\sinh(4u)$ in terms of x

(9***) Derive an expression for $\operatorname{arsinh}(a)$ in the form $\ln b$

(10*) Given that $\operatorname{artanh} x = \frac{1}{2} \ln \left(\frac{1+x}{1-x} \right)$ and $\operatorname{arcoth} x = \frac{1}{2} \ln \left(\frac{1+x}{x-1} \right)$,

and also that $\frac{d}{dx}(\operatorname{artanh} x) = \frac{d}{dx}(\operatorname{arcoth} x) = \frac{1}{1-x^2}$,

what is wrong with the following reasoning?

$$\int \frac{1}{1-x^2} dx = \operatorname{artanh} x + C = \operatorname{arcoth} x + C_1,$$

so that $\operatorname{artanh} x - \operatorname{arcoth} x = C_2$

$$\text{But } \operatorname{artanh} x - \operatorname{arcoth} x = \frac{1}{2} \ln \left(\frac{\left(\frac{1+x}{1-x} \right)}{\left(\frac{1+x}{x-1} \right)} \right) = \frac{1}{2} \ln \left(\frac{x-1}{1-x} \right) = \frac{1}{2} \ln(-1),$$

which isn't defined!

(11***) Given that $\sinh x = \tanh y$, where $-\frac{\pi}{2} < y < \frac{\pi}{2}$, show that

(a) $\tanh x = \sin y$ (b) $x = \ln(\tanh y + \sec y)$

(12***) What is the domain of $\operatorname{artanh}\left(\frac{x}{2}\right)$?

(13***) Show that $\operatorname{arcoth}x = \frac{1}{2}\ln\left(\frac{1+x}{x-1}\right)$ ($|x| > 1$)

(14***) (i) Use $\operatorname{artanh}x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$ to show that $\frac{d}{dx}\operatorname{artanh}x = \frac{1}{1-x^2}$

(ii) Use $\operatorname{arcoth}x = \frac{1}{2}\ln\left(\frac{1+x}{x-1}\right)$ to show that $\frac{d}{dx}\operatorname{arcoth}x = \frac{1}{1-x^2}$ also

(15***) (i) Show that $\operatorname{arcoth}x = \operatorname{artanh}\left(\frac{1}{x}\right)$

(ii) Find $f(x)$ such that $\operatorname{arcosh}x = \operatorname{arsinh}(f(x))$

(16*) Given that $\int \frac{1}{\sqrt{x^2-a^2}} dx = \operatorname{arcosh}\left(\frac{x}{a}\right)$, and that

$\operatorname{arcosh}x = \ln(x + \sqrt{x^2 - 1})$, justify the writing of the integral as $\ln(x + \sqrt{x^2 - a^2})$

(17***) Given that $\operatorname{arcosh}x = \ln(x + \sqrt{x^2 - 1})$, show that if

$\cosh a = b$ then $a = \ln(b \pm \sqrt{b^2 - 1})$

(18*) Sketch the following:

(i) $y = \operatorname{artanh}x$ (ii) $y = \operatorname{coth}x$ (iii) $y = \operatorname{arcoth}x$