Hyperbolic Functions - Exercises (43 pages ; 11/3/25)

(i) Prove, using exponential functions, that

(a) $cosh^2x - sinh^2x = 1$

(b) $sinh2x = 2sinhxcoshx$

(ii) By differentiating the result from (i)(b), obtain an expression for $cosh2x$ in terms of $cosh^2x$ and $sinh^2x$

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Solution

(i)(a) As $coshx = \frac{1}{2}(e^x + e^{-x})$ & $sinhx = \frac{1}{2}(e^x - e^{-x})$,

$cosh^2x - sinh^2x = (coshx + sinhx)(coshx - sinhx)$

$= e^x.e^{-x} = 1$

(b) $2sinhxcoshx = 2(\frac{1}{2})(e^x - e^{-x})(\frac{1}{2})(e^x + e^{-x})$

$= \frac{1}{2}(e^{2x} - e^{-2x}) = sinh2x$ (by difference of 2 squares)

(ii) Differentiating $sinh2x = 2sinhxcoshx$ gives

$2cosh2x = 2coshxcoshx + 2sinhxsinhx$

$\Rightarrow cosh2x = cosh^2x + sinh^2x$

(a) Find the formula connecting $tanh^2x$ & $sech^2x$?

(b) Find the formula connecting $coth^2x$ & co$sech^2x$?

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Solution

From $cosh^2x - sinh^2x = 1$,

(a) divide by $cosh^2x$, to give 1 - $tanh^2x = sech^2x$

(b) divide by $sinh^2x$, to give $coth^2x$ - 1 = co$sech^2x$

If $x = sinhu$, write $sinh(4u)$ in terms of x

[If $x = sinhu$, write $sinh(4u)$ in terms of x]

Solution

$sinh(4u) = 2\, sinh(2u)\, cosh(2u)$

$= 4 sinhucoshu(cosh^2u + sinh^2u)$

$= 4x\sqrt{1 + x^2}(1 + 2x^2)$

Given that $\int \frac{1}{\sqrt{x^2-a^2}} dx = arcosh(\frac{x}{a})$, and that

$arcoshx = ln\ (x + \sqrt{x^2 - 1})$, justify the writing of the integral as

$\ln\ (x + \sqrt{x^2 - a^2})$

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Solution

$arcosh\left(\frac{x}{a}\right) = l\,n\left(\frac{x}{a} + \sqrt{\frac{x^2}{a^2} - 1}\right) = \ln\ (\frac{x+\sqrt{x^2-a^2}}{a})$

$= \ln\left(x + \sqrt{x^2-a^2}\right) - lna$, which only differs from $\ln\left(x + \sqrt{x^2-a^2}\right)$ by a constant

Find or prove the following:

(i) $\frac{d}{dx}\ tanhx$

(ii) $\frac{d}{dx} arcoshx = \frac{1}{\sqrt{x^2-1}}$

(iii) $\frac{d}{dx}\ artanhx = \frac{1}{1-x^2}$

(iv) $\frac{d}{dx}\ sechx$

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Solutions

(i) $\frac{d}{dx}\ tanhx = \ \frac{d}{dx}\ \frac{sinhx}{coshx} = \ \frac{coshx.coshx - sinhx.sinhx}{cosh^2 x}$

$= \ \frac{1}{cosh^2 x} = sech^2 x$

(ii) Let $y = arcoshx$, so that $\cosh y = x$

Then $\frac{dx}{dy} = sinhy$ and $\frac{dy}{dx} = \frac{1}{\sqrt{cosh^2 y - 1}} = \ \frac{1}{\sqrt{x^2-1}}$

(iii) Let $y = artanhx$, so that $tanhy = x$

and $\frac{dx}{dy} = \ sech^2 y = 1 - tanh^2 y = 1 - x^2$

Hence $\frac{d}{dx}\ artanhx = \ \frac{1}{1-x^2}$

(iv) $\frac{d}{dx}\ sechx = \frac{d}{dx}\ (coshx)^{-1} = (-1)(coshx)^{-2} sinhx$

$= -sech^2 x. sinhx$ or $-sechx. tanhx$

Solve the equation $5cosh2x + 3sinhx = 6$,

giving your answers in exact logarithmic form.

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Solution

$5cosh2x + 3sinhx = 6 \Rightarrow 5(cosh^2x + sinh^2x) + 3sinhx - 6 = 0$

$\Rightarrow 5(1 + 2sinh^2x) + 3sinhx - 6 = 0$

$\Rightarrow 10sinh^2x + 3sinhx - 1 = 0$

$\Rightarrow (5sinhx - 1)(2sinhx + 1) = 0$

$\Rightarrow sinhx = \frac{1}{5}$ or $-\frac{1}{2}$

$\Rightarrow x = arsinh(\frac{1}{5})$ or $arsinh(-\frac{1}{2})$

$\Rightarrow x = \ln\,(\frac{1}{5} + \sqrt{\frac{1}{25} + 1})$ or $\ln\,(-\frac{1}{2} + \sqrt{\frac{1}{4} + 1})$

or $\ln\,(\frac{1}{5}\left(1 + \sqrt{26}\right))$ or $\ln\,(\frac{1}{2}\left(\sqrt{5} - 1\right))$

[It is possible to substitute these values into the equation, as a check.]

Using the logarithmic form of $arcoshx$

$(arcoshx = ln\,(x + \sqrt{x^2 - 1})$, prove that the derivative of

$arcoshx$ is $\frac{1}{\sqrt{x^2-1}}$

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Solution

$\frac{d}{dx} ln\left(x + \sqrt{x^2 - 1}\right) = \frac{1}{x+\sqrt{x^2-1}} \ .(1 + \frac{1}{2}(x^2 - 1)^{-\frac{1}{2}}(2x))$

$= \frac{1}{x+\sqrt{x^2-1}} \ .\frac{1}{\sqrt{x^2-1}} \ (\sqrt{x^2 - 1} + x)$

$= \frac{1}{\sqrt{x^2-1}}$

Show that $artanhx = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \quad (|x| < 1)$

[Show that $artanhx = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \quad (|x| < 1)$]

Solution

If $y = artanhx$, then $tanhy = x \quad (|x| < 1)$

$\Rightarrow x = \frac{\frac{1}{2}(e^y - e^{-y})}{\frac{1}{2}(e^y + e^{-y})} = \frac{e^{2y}-1}{e^{2y}+1}$

$\Rightarrow x(e^{2y} + 1) = e^{2y} - 1$

$\Rightarrow e^{2y}(x - 1) = -1 - x$

$\Rightarrow e^{2y} = \frac{1+x}{1-x}$

$\Rightarrow y = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \quad (|x| < 1)$

Show that $arcoshx = \ln\left(x + \sqrt{x^2 - 1}\right)$

[Show that $arcoshx = \ln(x + \sqrt{x^2 - 1})$]

Solution

If $y = arcoshx$, then $coshy = x$

$\Rightarrow \; x = \frac{1}{2}(e^y + e^{-y})$

$\Rightarrow 2xe^y = e^{2y} + 1$

$\Rightarrow \; e^{2y} - 2xe^y + 1 = 0$

$\Rightarrow e^y = \frac{2x \pm \sqrt{4x^2 - 4}}{2} = x \pm \sqrt{x^2 - 1}$

$\Rightarrow y = ln\left(x \pm \sqrt{x^2 - 1}\right)$

However, in order for $y = arcoshx$ to be a function, the negative branch is suppressed (by restricting the domain of $coshx$ to non-negative values). And we can show that $ln\left(x - \sqrt{x^2 - 1}\right) < 0$:

Method 1

Equivalently, we need to show that $x - \sqrt{x^2 - 1} < 1$; or that

$x - 1 < \sqrt{x^2 - 1}$

But $x - 1 = \sqrt{(x-1)(x-1)}$ (noting that the range of $coshx$, and hence the domain of $arcoshx$, excludes $x < 1$)

and $\sqrt{(x-1)(x-1)} < \sqrt{(x-1)(x+1)} = \sqrt{x^2 - 1}$, as required

($y = \sqrt{x}$ is an increasing function,

so $x - 1 < x + 1 \Rightarrow \sqrt{x-1} < \sqrt{x+1}$)

[Alternatively, we can argue (slightly informally) that the difference between x^2 and $x^2 - 1$ (ie 1) is contracted by applying the square root function, so that $\sqrt{x^2} - \sqrt{x^2 - 1} < 1$]

Method 2

We expect the unrestricted $y = arcoshx$ to be symmetric about the x-axis (as $y = coshx$ is symmetric about the y-axis). So we could show that $y = ln\left(x \pm \sqrt{x^2 - 1}\right)$ can also be written as

$y = \pm ln\left(x + \sqrt{x^2 - 1}\right)$, and then reject the negative branch as before.

So we want to show that $ln\left(x - \sqrt{x^2 - 1}\right) = -ln\left(x + \sqrt{x^2 - 1}\right)$:

RHS $= ln\left(\frac{1}{x+\sqrt{x^2-1}}\right) = ln\left(\frac{x-\sqrt{x^2-1}}{x^2-(x^2-1)}\right) = ln\left(x - \sqrt{x^2 - 1}\right)$, as required

Derive an expression for $arsinh(a)$ in the form lnb

[Derive an expression for $arsinh(a)$ in the form lnb]

Solution

Let $x = arsinh(a)$, so that $sinhx = a$

and $\frac{1}{2}(e^x - e^{-x}) = a$

Then $\frac{1}{2}(e^{2x} - 1) = ae^x$

and $e^{2x} - 2ae^x - 1 = 0$,

so that $e^x = \frac{2a \pm \sqrt{4a^2+4}}{2} = a + \sqrt{a^2+1}$ (rejecting the negative root)

Thus $arsinh(a) = \ln\left(a + \sqrt{a^2+1}\right)$

(noting that $a + \sqrt{a^2+1} > 0$)

[Note that $arsinh\left(\frac{x}{a}\right) = \ln\left(\frac{x}{a} + \sqrt{\left(\frac{x}{a}\right)^2 + 1}\right)$

$= \ln\left(\frac{x+\sqrt{x^2+a^2}}{a}\right) = \ln\left(x + \sqrt{x^2+a^2}\right) - lna$

In the formulae booklets, $\int \frac{1}{\sqrt{a^2+x^2}} dx$ is often given as

"$arsinh\left(\frac{x}{a}\right)$ or $\ln\left(x + \sqrt{x^2+a^2}\right)$" but, as we've just seen, these two expressions differ by a constant.]

What is the domain of $artanh\left(\frac{x}{2}\right)$?

[What is the domain of $artanh\left(\frac{x}{2}\right)$?]

Solution

$$tanhx = \frac{sinhx}{coshx} = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^{2x} - 1}{e^{2x} + 1} = \frac{e^{2x} + 1}{e^{2x} + 1} - \frac{2}{e^{2x} + 1} = 1 - \frac{2}{e^{2x} + 1}$$

Thus $-1 < tanhx < 1$ (as $x \to -\infty$ & ∞)

As $artanhx$ is the inverse of $tanhx$, the domain of $artanhx$ is the range of $tanhx$; ie $(-1, 1)$.

Thus the domain of $artanh\left(\frac{x}{2}\right)$ satisfies $-1 < \frac{x}{2} < 1$;

ie $-2 < x < 2$

Show that $arcothx = \frac{1}{2} ln\left(\frac{1+x}{x-1}\right) \quad (|x| > 1)$

[Show that $arcothx = \frac{1}{2}ln\left(\frac{1+x}{x-1}\right) \quad (|x| > 1)$]

Solution

If $y = arcothx$, then $cothy = x \quad (|x| > 1)$

$\Rightarrow x = \frac{\frac{1}{2}(e^y+e^{-y})}{\frac{1}{2}(e^y-e^{-y})} = \frac{e^{2y}+1}{e^{2y}-1}$

$\Rightarrow x(e^{2y} - 1) = e^{2y} + 1$

$\Rightarrow e^{2y}(x - 1) = 1 + x$

$\Rightarrow e^{2y} = \frac{1+x}{x-1}$

$\Rightarrow y = \frac{1}{2}\ln\left(\frac{1+x}{x-1}\right) \quad (|x| > 1)$

Alternative Method

If $artanhx = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \quad (|x| < 1)$ has been established:

If $y = arcothx$, then $cothy = x$

$\Rightarrow tanhy = \frac{1}{x}$,

and hence $y = artanh\left(\frac{1}{x}\right) = \frac{1}{2}ln\left(\frac{1+\frac{1}{x}}{1-\frac{1}{x}}\right) = \frac{1}{2}ln\left(\frac{x+1}{x-1}\right)$

(i) Use $artanhx = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$ to show that $\frac{d}{dx}artanhx = \frac{1}{1-x^2}$

(ii) Use $arcothx = \frac{1}{2}\ln\left(\frac{1+x}{x-1}\right)$ to show that $\frac{d}{dx}arcothx = \frac{1}{1-x^2}$ also [(i) Use $artanhx = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$ to show that $\frac{d}{dx}artanhx = \frac{1}{1-x^2}$

(ii) Use $arcothx = \frac{1}{2}\ln\left(\frac{1+x}{x-1}\right)$ to show that $\frac{d}{dx}arcothx = \frac{1}{1-x^2}$ also]

Solution

(i) $\frac{d}{dx}artanhx = \frac{1}{2}.\frac{1-x}{1+x}.\frac{(1-x)-(1+x)(-1)}{(1-x)^2}$

$= \frac{1}{2}.\frac{2}{(1+x)(1-x)} = \frac{1}{1-x^2}$

(ii) $\frac{d}{dx}arcothx = \frac{1}{2}.\frac{x-1}{1+x}.\frac{(x-1)-(1+x)}{(x-1)^2}$

$= \frac{1}{2}.\frac{-2}{(1+x)(x-1)} = \frac{1}{1-x^2}$

(i) Show that $arcothx = artanh\left(\frac{1}{x}\right)$

(ii) Find $f(x)$ such that $arcoshx = arsinh\big(f(x)\big)$

[(i) Show that $arcothx = artanh\left(\frac{1}{x}\right)$

(ii) Find $f(x)$ such that $arcoshx = arsinh(f(x))$]

Solution

(i) Let $y = arcothx$, so that $\coth y = x$

$\Rightarrow tanhy = \frac{1}{x}$

$\Rightarrow y = artanh\left(\frac{1}{x}\right)$

(ii) Let $y = arcoshx$, so that $coshy = x$

$\Rightarrow sinhy = \sqrt{x^2 - 1}$

$\Rightarrow y = arsinh\left(\sqrt{x^2 - 1}\right)$; ie $f(x) = \sqrt{x^2 - 1}$

Given that $arcoshx = ln(x + \sqrt{x^2 - 1})$, show that if

$cosha = b$ then $a = \ln\,(b \pm \sqrt{b^2 - 1})$

[Given that $arcoshx = ln(x + \sqrt{x^2 - 1})$, show that if

$cosha = b$ then $a = \ln{(b \pm \sqrt{b^2 - 1})}$]

Solution

$cosha = b \Rightarrow a = \pm arcoshb = \pm ln(b + \sqrt{b^2 - 1})$

And $-ln\left(b + \sqrt{b^2 - 1}\right) = \ln\left(\frac{1}{b+\sqrt{b^2-1}}\right) = ln(\frac{b-\sqrt{b^2-1}}{b^2-(b^2-1)})$

$= ln(b - \sqrt{b^2 - 1})$

so that $\pm ln\left(b + \sqrt{b^2 - 1}\right) = ln\ (b \pm \sqrt{b^2 - 1})$

Write $ln a$ in the form $arsinh\left(f(a)\right)$, where $f(a)$ is some expression in terms of a.

[Write lna in the form $arsinh\left(f(a)\right)$, where $f(a)$ is some expression in terms of a.]

Solution

Writing $lna = arsinh(b)$,

$sinh(lna) = b$, and so $b = \frac{1}{2}(e^{lna} - e^{-lna}) = \frac{1}{2}(a - \frac{1}{a})$,

and thus $lna = arsinh[\frac{1}{2}\left(a - \frac{1}{a}\right)]$, for $a > 0$

Simplify sinh $(cosh^{-1}2)$

[Simplify sinh ($cosh^{-1}2$)]

Solution

Let $cosh^{-1}2 = a(> 0)$,

so that $2 = cosha$

Then $sinha = +\sqrt{cosh^2a - 1}$ [as a > 0]

$= \sqrt{3}$

Given that $artanhx = \frac{1}{2}ln\left(\frac{1+x}{1-x}\right)$ and $arcothx = \frac{1}{2}ln\left(\frac{1+x}{x-1}\right)$,

and also that $\frac{d}{dx}(artanhx) = \frac{d}{dx}(arcothx) = \frac{1}{1-x^2}$,

what is wrong with the following reasoning?

$\int \frac{1}{1-x^2}\,dx = artanhx + C = arcothx + C_1$,

so that $artanhx - arcothx = C_2$

But $artanhx - arcothx = \frac{1}{2}ln\left(\frac{\left(\frac{1+x}{1-x}\right)}{\left(\frac{1+x}{x-1}\right)}\right) = \frac{1}{2}ln\left(\frac{x-1}{1-x}\right) = \frac{1}{2}\ln(-1)$,

which isn't defined!

[Given that $artanhx = \frac{1}{2}ln\left(\frac{1+x}{1-x}\right)$ and $arcothx = \frac{1}{2}ln\left(\frac{1+x}{x-1}\right)$,

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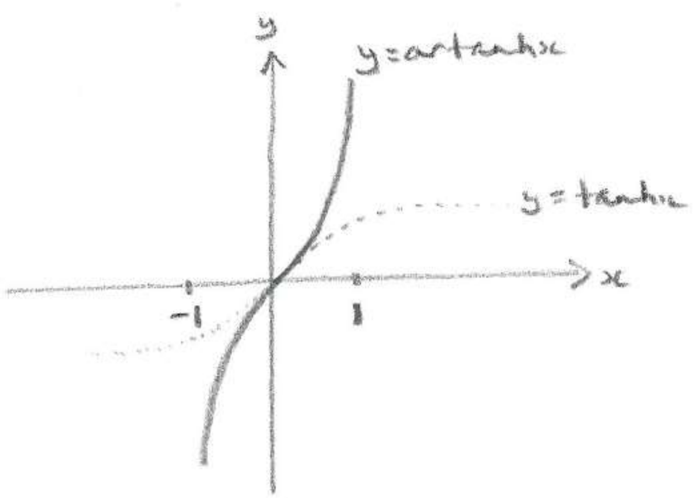
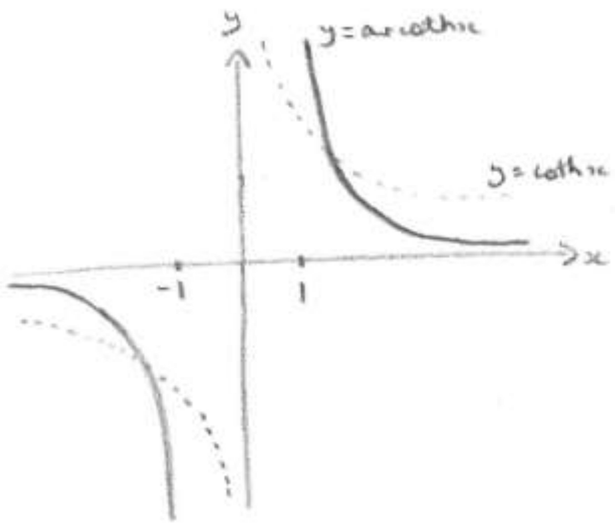
which isn't defined!]

Solution

The problem is that the domains of $y = artanhx$ and $y = arcothx$ don't overlap (see graphs below). We ought to say that $artanhx = \frac{1}{2}ln\left(\frac{1+x}{1-x}\right)$ for $|x| < 1$ and $arcothx = \frac{1}{2}ln\left(\frac{1+x}{x-1}\right)$ for $|x| > 1$. So it doesn't make sense to determine $artanhx - arcothx$





Note that, with $|x| < 1$, $\frac{d}{dx}(artanhx) = \frac{1}{1-x^2} > 0$ for all x; whilst with $|x| > 1$, $\frac{d}{dx}(arcothx) = \frac{1}{1-x^2} < 0$ for all x

Simplify $sinh(arcoshx)$ & cosh $(arsinhx)$

[Simplify $sinh(arcoshx)$ & $cosh(arsinhx)$]

Solution

$sinh(arcoshx) = sinh\ (ln\ (x + \sqrt{x^2 - 1})$

$= \frac{1}{2}([x + \sqrt{x^2 - 1}] - [\frac{1}{x+\sqrt{x^2-1}}])$

$= \frac{1}{2}([x + \sqrt{x^2 - 1}] - [\frac{1}{x+\sqrt{x^2-1}}][\frac{x-\sqrt{x^2-1}}{x-\sqrt{x^2-1}}])$

$= \frac{1}{2}([x + \sqrt{x^2 - 1}] - \frac{[x-\sqrt{x^2-1}]}{1})$

$= \sqrt{x^2 - 1}$

$cosh(arsinhx) = cosh\ (ln\ (x + \sqrt{x^2 + 1})$

$= \frac{1}{2}([x + \sqrt{x^2 + 1}] + [\frac{1}{x+\sqrt{x^2+1}}])$

$= \frac{1}{2}([x + \sqrt{x^2 + 1}] + [\frac{1}{x+\sqrt{x^2+1}}][\frac{x-\sqrt{x^2+1}}{x-\sqrt{x^2+1}}])$

$= \frac{1}{2}([x + \sqrt{x^2 + 1}] + \frac{[x-\sqrt{x^2+1}]}{-1})$

$= \sqrt{x^2 + 1}$

Given that $sinhx = tany$, where $-\frac{\pi}{2} < y < \frac{\pi}{2}$, show that

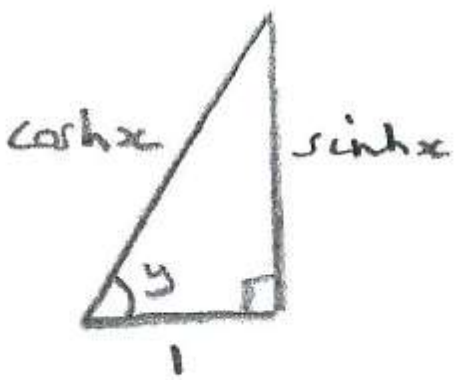
(a) $tanhx = siny$ (b) $x = \ln(tany + secy)$

[Given that $sinhx = tany$, where $-\frac{\pi}{2} < y < \frac{\pi}{2}$, show that

(a) $tanhx = siny$ (b) $x = \ln{(tany + secy)}$]

Solution

(a) As $sinhx = tany$, we can construct a right-angled triangle (see diagram below), where the hypotenuse is $coshx$, as $sinh^2x + 1 = cosh^2x$.



Then $siny = \frac{sinhx}{coshx} = tanhx$, as required.

Alternatively: $tanhx = \frac{sinhx}{coshx} = \frac{tany}{\sqrt{1+sinh^2x}}$

(from $sinh^2x + 1 = cosh^2x$, noting that $coshx$ is always positive, so that we take the positive square root)

$= \frac{tany}{\sqrt{1+tan^2y}} = \frac{tany}{\sqrt{sec^2y}} = \frac{tany}{secy}$

(as $cosy > 0$ when $-\frac{\pi}{2} < y < \frac{\pi}{2}$, and hence $secy > 0$ also)

$= tanycosy = siny$

(b) From the right-angled triangle,

$tany + secy = sinhx + coshx$

$= \frac{1}{2}(e^x - e^{-x}) + \frac{1}{2}(e^x + e^{-x}) = e^x$,

so that $\ln(tany + secy) = x$, as required.

Alternatively: $sinhx = tany \Rightarrow \frac{1}{2}(e^x - e^{-x}) = tany$

$\Rightarrow e^{2x} - 1 = 2tanye^x$

$\Rightarrow e^{2x} - 2tanye^x - 1 = 0$

$\Rightarrow e^x = \frac{2tany \pm \sqrt{4tan^2y + 4}}{2} = tany \pm secy$

$tany - secy = \frac{siny - 1}{cosy} < 0$ when $-\frac{\pi}{2} < y < \frac{\pi}{2}$

Hence, as $e^x > 0$, it follows that $e^x = tany + secy$,

and hence $x = \ln(tany + secy)$, as required.