Hyperbolas - Exercises (Solutions) (6 pages; 30/12/19)

(1**) Show that the equation of the tangent to the hyperbola

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point $(acosht, bsinht)$ is

$yasinht = xbcosht - ab$

Solution

Using the parametric equations $x = acosht$ & $y = bsinht$,

$\frac{dx}{dt} = asinht$ & $\frac{dy}{dt} = bcosht$,

so that $\frac{dy}{dx} = \frac{bcosht}{asinht}$

and the equation of the tangent at $(acosht, bsinht)$ is

$\frac{y-bsinht}{x-acosht} = \frac{bcosht}{asinht}$

and hence $yasinht - absinh^2t = xbcosht - abcosh^2t$,

so that $yasinht = xbcosht - ab$

(2***) Given that the tangent in (1) meets the asymptotes of the hyperbola at the points P & Q, show that the mid-point of P & Q is $(acosht, bsinht)$.

Solution

The asymptotes of the hyperbola are $y = \pm\frac{b}{a}x$

From (1), the tangent to the hyperbola at $(acosht, bsinht)$ meets the asymptote $y = \frac{b}{a}x$ at P (say), where $bxsinht = xbcosht - ab$ and the asymptote $y = -\frac{b}{a}x$ at Q where

$-bxsinht = xbcosht - ab$

so that P is the point $\left(\frac{a}{cosht-sinht}, \frac{b}{cosht-sinht}\right)$

and Q is the point $\left(\frac{a}{cosht+sinht}, \frac{-b}{cosht+sinht}\right)$

The mid-point of P & Q is therefore

$$\left(\frac{1}{2}\left[\frac{a}{cosht-sinht}+\frac{a}{cosht+sinht}\right], \frac{1}{2}\left[\frac{b}{cosht-sinht}+\frac{-b}{cosht+sinht}\right]\right)$$

$$= \left(\frac{acosht}{cosh^2t-sinh^2t}, \frac{bsinht}{cosh^2t-sinh^2t}\right) = (acosht, bsinht)$$, as required.

(3***) In the case where $b = a$, find the area of the triangle OPQ (where P & Q are as in (2), and O is the Origin).

Solution

The two asymptotes are at right angles to each other, so that the required area, $A = \frac{1}{2}OP.OQ$

Then $4A^2 = \left(\left(\frac{a}{cosht-sinht}\right)^2 + \left(\frac{a}{cosht-sinht}\right)^2\right)$

$$\times\left(\left(\frac{a}{cosht+sinht}\right)^2 + \left(\frac{-a}{cosht+sinht}\right)^2\right)$$

$$= \left(\frac{2a^2}{(cosht-sinht)^2}\right)\left(\frac{2a^2}{(cosht+sinht)^2}\right)$$

$$= \frac{4a^4}{(cosh^2t-sinh^2t)^2} = 4a^4$$

and therefore $A = a^2$

(4***) The chord PQ, where P and Q are points on the rectangular hyperbola $xy = c^2$, has gradient 1. Show that the locus of the point of intersection of the tangents from P and Q is the line

$y = -x$. [Edx FP3 textbook, Ex. 2G, Q9]

Solution

Let P & Q be the points $\left(ct_1, \frac{c}{t_1}\right)$ & $\left(ct_2, \frac{c}{t_2}\right)$, respectively.

As the gradient of PQ is 1, $\frac{\frac{c}{t_2}-\frac{c}{t_1}}{ct_2-ct_1} = 1$, so that

$\frac{1}{t_2} - \frac{1}{t_1} = t_2 - t_1$

$\Rightarrow \frac{t_1-t_2}{t_1t_2} = t_2 - t_1$

$\Rightarrow t_1t_2 = -1$, as $t_1 \neq t_2$ (P & Q being distinct points)

The equation of the tangent from P is:

$\frac{y-\frac{c}{t_1}}{x-ct_1} = \frac{dy/dt}{dx/dt} |t = t_1$, where $x = ct$ & $y = \frac{c}{t}$

so that $dy/dt = -\frac{c}{t^2}$ & $dx/dt = c$

and the equation of the tangent from P is

$\frac{y-\frac{c}{t_1}}{x-ct_1} = \frac{\left(-\frac{c}{t_1{}^2}\right)}{c} \Rightarrow t_1{}^2y - t_1c = -(x - ct_1)$

$\Rightarrow t_1{}^2y = -x + 2ct_1$ (1)

Similarly, the equation of the tangent from Q is $t_2{}^2y = -x + 2ct_2$

and these lines intersect where

$t_1{}^2y - 2ct_1 = t_2{}^2y - 2ct_2$,

so that $y(t_1{}^2 - t_2{}^2) = 2c(t_1 - t_2)$

and $y = \frac{2c}{t_1+t_2}$ (as $t_1 \neq t_2$)

Then, from (1), $x = 2ct_1 - \frac{2ct_1{}^2}{t_1+t_2}$

$= \frac{2ct_1{}^2+2ct_1t_2-2ct_1{}^2}{t_1+t_2}$

$= \frac{2ct_1t_2}{t_1+t_2}$

and so $\frac{y}{x} = \frac{1}{t_1t_2} = -1$ (found earlier),

and thus the points of intersection satisfy $y = -x$, as required.

(5***) Use matrices to show that the rectangular hyperbola $x^2 - y^2 = a^2$ can be obtained by rotating the rectangular hyperbola $xy = c^2$, expressing a^2 in terms of c.

Solution

The asymptotes of $x^2 - y^2 = a^2$ are $y = \pm x$, whilst the asymptotes of

$xy = c^2$ are the x and y axes.

So consider a rotation of 45° clockwise.

Then the point (x, y) on the hyperbola $xy = c^2$ is transformed to the point (u, v), where

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} cos(-45°) & -sin(-45°) \\ sin(-45°) & cos(-45°) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} x + y \\ -x + y \end{pmatrix}$$

Then $u^2 - v^2 = (u - v)(u + v)$

$= \frac{1}{\sqrt{2}}(2x).\frac{1}{\sqrt{2}}(2y) = 2xy = 2c^2$

Relabelling gives $x^2 - y^2 = 2c^2$ (and $a^2 = 2c^2$).

(6**) Show that the equation of the normal to the hyperbola

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point $(acosht, bsinht)$ is

$xasinht + ybcosht = (a^2 + b^2)sinhtcosht$

Solution

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{bcosht}{asinht}$

so that equation of normal is $y - bsinht = -\frac{asinht}{bcosht}(x - acosht)$

$\Rightarrow bcosht.y - b^2sinhtcosht = -xasinht + a^2sinhtcosht$

$\Rightarrow xasinht + ybcosht = (a^2 + b^2)sinhtcosht$, as required

(7***) [Edx (Pearson), FP1, ME2, Q7 (p58)]

l_1& l_2 are distinct tangents to the rectangular hyperbola $xy = 9$ with gradient $-\frac{1}{4}$; find the equations of l_1& l_2

Solution

$y = \frac{9}{x} \Rightarrow \frac{dy}{dx} = -9x^{-2}$

Suppose that the gradient at the point $(a, \frac{9}{a})$ is $-\frac{1}{4}$

Then $-9a^{-2} = -\frac{1}{4}$, so that $a^2 = 36$, and $a = \pm 6$

Equation of l_1: $y - \frac{9}{6} = -\frac{1}{4}(x - 6)$;

or $4y - 6 = -x + 6$; $x + 4y = 12$

Equation of l_2: $y - (-\frac{9}{6}) = -\frac{1}{4}(x - [-6])$;

or $4y + 6 = -x - 6$; $x + 4y = -12$

(8***) [Edx (Pearson), FP1, ME2, Q14 (p59)]

Suppose that P is a general point on a rectangular hyperbola and that the tangent at P crosses the x and y axes at A and B respectively. Show that:

(i) $AP = BP$

(ii) the triangle OAB has a constant area, as P varies

Solution

(i) Suppose that the equation of the rectangular hyperbola is $xy = c^2$ and that P is the point $\left(a, \frac{c^2}{a}\right)$.

Then $\frac{dy}{dx} = -c^2x^{-2}$, and the equation of the tangent at P is

$y - \frac{c^2}{a} = -\frac{c^2}{a^2}(x - a)$

At A, $0 - \frac{c^2}{a} = -\frac{c^2}{a^2}(x - a)$, so that $a = x - a$ and $x = 2a$

At B, $y - \frac{c^2}{a} = -\frac{c^2}{a^2}(0 - a)$, so that $y = \frac{c^2}{a} + \frac{c^2}{a} = \frac{2c^2}{a}$

Then $AP^2 = (2a - a)^2 + (\frac{c^2}{a} - 0)^2 = a^2 + \frac{c^4}{a^2}$

and $BP^2 = (a - 0)^2 + (\frac{2c^2}{a} - \frac{c^2}{a})^2 = a^2 + \frac{c^4}{a^2}$

Thus $AP = BP$, as required.

(ii) Area of OAB $= \frac{1}{2}(2a)\left(\frac{2c^2}{a}\right) = 2c^2$; ie a constant