Hooke's Law: Ideas & Exercises (15 pages; 5/2/25)

(1) The tension in a string or spring, $T = \frac{\lambda x}{l}$, where λ is the modulus of elasticity, x is the extension, and l is the original length.

[Note: In the case of a spring, x could be negative, giving a negative tension (ie a compression).]

Alternatively, $T = kx$, where k is the stiffness.

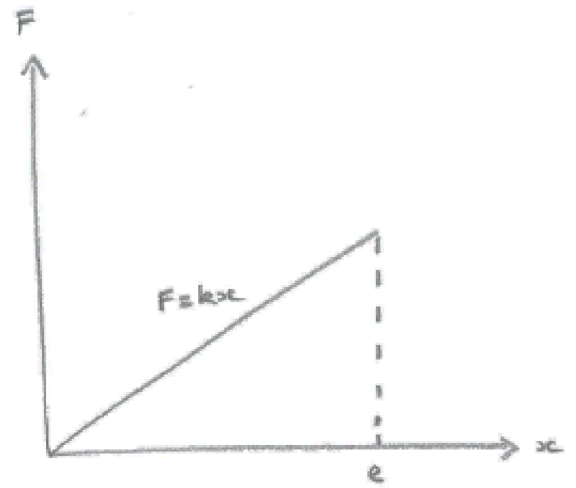
[See Appendix for more details.]

(2) Elastic Potential Energy

Work done by the tension in stretching a string or spring to an extension e is:

$\int_0^e \frac{\lambda x}{l} dx = \frac{\lambda}{l}\left[\frac{1}{2}x^2\right]_0^e = \frac{\lambda e^2}{2l}$ (or $\frac{1}{2}ke^2$)

This is the area under the Force-displacement graph:



A horizontal spring has modulus of elasticity 200N and natural length 20cm. Find the work done in stretching it from a length of 25cm to a length of 30cm.

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Solution

Gain in elastic PE $= \frac{1}{2}\left(\frac{200}{0.2}\right)(0.1)^2 - \frac{1}{2}\left(\frac{200}{0.2}\right)(0.05)^2$

$= 500(0.01 - 0.0025) = 3.75\ J$

(3) A particle of mass 200g hangs at a point Q, suspended from a fixed point P, by means of a spring of original length 20cm and modulus of elasticity 5N.

Find the distance PQ

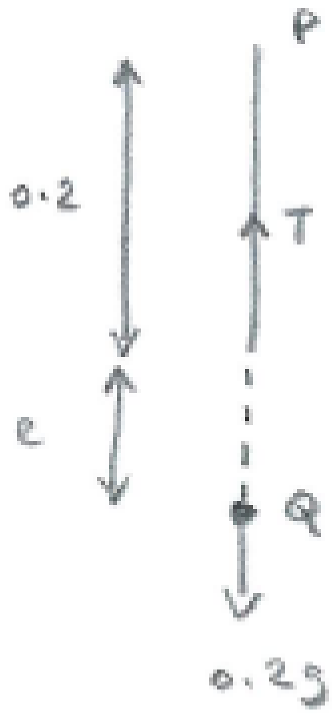
[A particle of mass 200g hangs at a point Q, suspended from a fixed point P, by means of a spring of original length 20cm and modulus of elasticity 5N.]

Hooke's Law $\Rightarrow \frac{\lambda e}{l} = T$

$N2L \Rightarrow T = mg$

Hence $\frac{5e}{0.2} = (0.2)(9.8) \Rightarrow e = 0.0784$

$\Rightarrow PQ = 0.2784$; ie 27.84 cm



The particle is then pulled down to a point R, which is 35cm below P. The particle is then released. There is no resistance to motion.

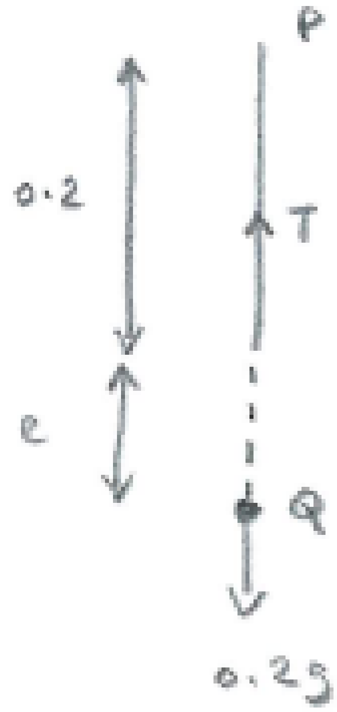
Find the acceleration of the particle when it is released from R.

[A particle of mass 200g hangs at a point Q, suspended from a fixed point P, by means of a spring of original length 20cm and modulus of elasticity 5N. The particle is then pulled down to a point R, which is 35cm below P. The particle is then released.
$PQ = 0.2784$]

Hooke's Law $\Rightarrow T = \frac{(5)(0.15)}{0.2} = 3.75\ N$

N2L $\Rightarrow 3.75 - (0.2)(9.8) = (0.2)a$ (taking upwards as positive)

$\Rightarrow a = 8.95ms^{-2}$ upwards



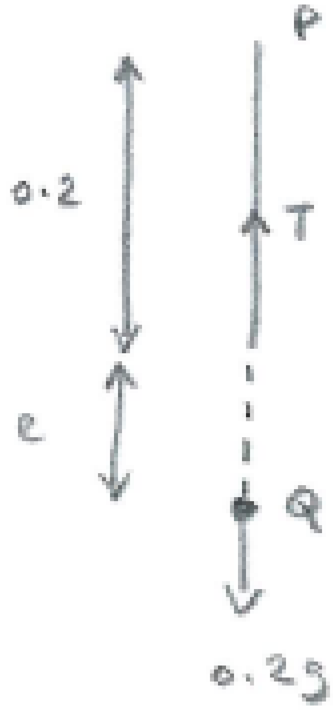
Find the work done in pulling the particle down to R

[A particle of mass 200g hangs at a point Q, suspended from a fixed point P, by means of a spring of original length 20cm and modulus of elasticity 5N. The particle is then pulled down to a point R, which is 35cm below P. The particle is then released. $PQ = 0.2784$]

Total energy of the particle at Q is

$(0.2)(9.8)(0.35 - 0.2 - 0.0784) + \frac{1}{2}\left(\frac{5}{0.2}\right)(0.0784)^2 + 0$

$= 0.140336 + 0.076832 + 0 = 0.217168$



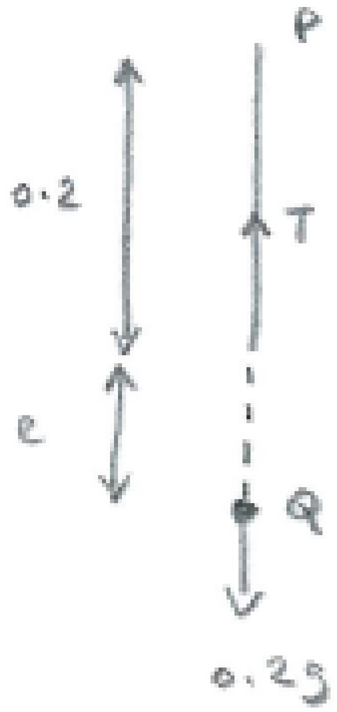
Total energy of the particle at R is:

$0 + \frac{1}{2}\left(\frac{5}{0.2}\right)(0.15)^2 + 0 = 0.28125$

Thus the work done $= 0.28125 - 0.217168 = 0.064082 = 0.0641$ J (3sf)

Find the maximum speed of the particle after it is released, and the point at which this occurs.

[A particle of mass 200g hangs at a point Q, suspended from a fixed point P, by means of a spring of original length 20cm and modulus of elasticity 5N. The particle is then pulled down to a point R, which is 35cm below P. The particle is then released. $PQ = 0.2784$]



The maximum speed will occur when the particle is not accelerating; ie at Q, where the net force on the particle is zero.

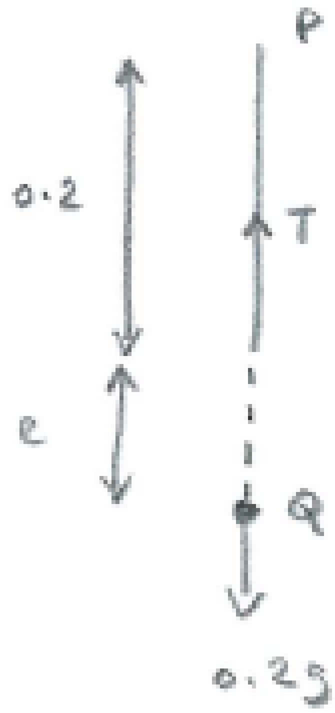
The KE of the particle at Q will equal the work done to pull it down to R, as this is the energy gained by the particle since it was last at Q.

Hence $\frac{1}{2}(0.2)v^2 = 0.064082$ (where v is the maximum speed)

and $v = 0.80051 = 0.801\ ms^{-1}$ (3sf)

Find the distance of the particle below P when it reaches its maximum height, at position S, and show that the distance QS equals the distance QR.

[A particle of mass 200g hangs at a point Q, suspended from a fixed point P, by means of a spring of original length 20cm and modulus of elasticity 5N. The particle is then pulled down to a point R, which is 35cm below P. The particle is then released. $PQ = 0.2784$]



Let d be the distance below P when the particle is at S.

The total energy of the particle at S is:

$(0.2)(9.8)(0.35 - d) + \frac{1}{2}\left(\frac{5}{0.2}\right)(d - 0.2)^2 + 0$

and this equals the energy at R of 0.28125 ,

so that $12.5d^2 - 6.96d + 0.90475 = 0$

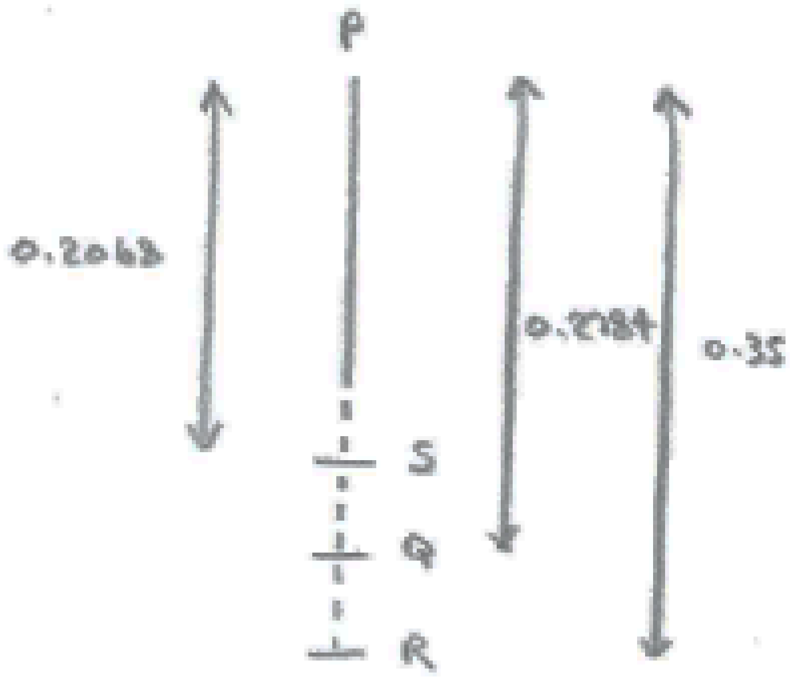
and $d = \frac{6.96 \pm \sqrt{3.2041}}{25} = 0.35 \; or \; 0.2068$

Thus 0.35 corresponds to R and S is the point 20.68cm below P.

This is $20 + 7.84 - 20.68 = 7.16$ cm above the equilibrium position Q,

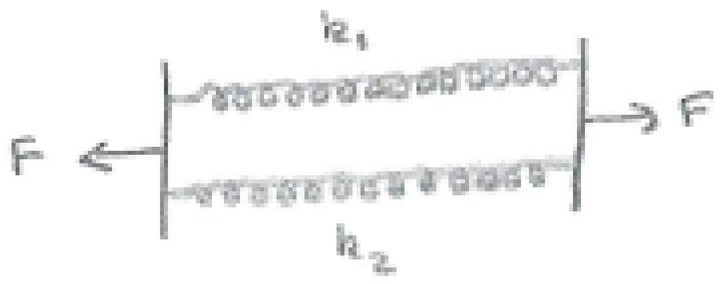
whilst R is $35 - (20 + 7.84) = 7.16$ cm below Q.

[The particle oscillates between R and S.]



(4) Springs in parallel

Consider two springs of the same original length and stiffnesses $k_1 \& k_2$, held in equilibrium in parallel, as shown below.



(Note: This system only makes sense if the original lengths are the same, so that when no force is applied, the springs both reach to the two sides.)

Let the stiffness of the combined springs be k, with extension e.

Show that $k = k_1 + k_2$.

Solution

Let T_1 & T_2 be the tensions in the two springs.

Then $F = ke$, $T_1 = k_1e$, $T_2 = k_2e,$ and $F = T_1 + T_2$

Hence $ke = k_1e + k_2e$, and so $k = k_1 + k_2$

(5) Springs in series

Consider two springs of stiffness $k_1 \& k_2$, held in equilibrium in series, as shown below.

By drawing separate force diagrams for the two springs, show that if k is the stiffness of the combined springs, then $\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2}$

Solution

By N2L, $T_1 = F$ and $T_2 = F$

Also, $T_1 = T_2$, by N3L.

By Hooke's Law, $F = k_1 e_1$ & $F = k_2 e_2$,

where e_1& e_2 are the extensions of the two springs.

Also, $F = k(e_1 + e_2)$

and so $k = \frac{F}{e_1+e_2} = \frac{F}{\frac{F}{k_1}+\frac{F}{k_2}}$

$\Rightarrow \frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2}$

Note: This can be extended to more than two springs, by replacing $\frac{1}{k_2}$ with $\frac{1}{k_3} + \frac{1}{k_4}$, and then re-labelling to give $\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3}$

Appendix

Hooke's law can be expressed in 3 ways:

(i) $T = \frac{EAx}{l}$, where E is Young's modulus and A is the cross-sectional area of the string or spring

E depends only on the nature of the material, and has S.I. units of Nm^{-2} or Pascals (Pa).

(ii) $T = \frac{\lambda x}{l}$, where λ is the modulus of elasticity of the string or spring.

As well as the nature of the material, λ depends on the cross-sectional area of the string or spring. Its S.I. units are N.

(iii) $T = kx$, where k is the stiffness of the string or spring.

As well as the nature of the material and the cross-sectional area of the string, k depends on the original length of the string or spring. In other words, k is specific to each piece of string or spring. Its S.I. units are Nm^{-1}.