Groups - Exercises (2 pages; 20/2/20)

Key to difficulty:

* easier

** moderate

*** harder

(1***) Multiplication modulo m (or just mod m), denoted by $\times_m$, is defined on the set $\{0,1,2, \dots , m-1\}$ by carrying out ordinary multiplication and taking the remainder when the product is divided by m. For example, $5 \times_3 4 = 2$.

Show that $\times_5$ is a closed and commutative binary operation on the set $\{0,1,2, \dots ,4\}$, and identify the inverse of each element, where it exists.

(2***)(i) Show that the set $\{1,4,7,13\}$ forms a group, under multiplication modulo 15.

(ii) Find the generators of the group.

(iii) Establish whether the group is cyclic.

(iv) Identify all the subgroups.

(3***) For the group $\left\{x, 1-x, \frac{1}{x}, \frac{1}{1-x}, \frac{x-1}{x}, \ \frac{x}{x-1}\right\}$ under composition of functions, where $x \in \mathbb{R}, x \neq 0,1$:

(i) Establish whether the group is abelian.

(ii) Find the periods of the elements of the group, and hence identify its proper subgroups.

(4***) Establish which of the following groups are isomorphic to each other:

(i) {0,1,2,3} ; addition modulo 4

(ii) {1,2,4,8} ; multiplication modulo 15

(iii) {3,6,9,12} ; multiplication modulo 15

(iv) {1,3,5,7} ; multiplication modulo 8

(v) $\{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}, \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\}$; matrix multiplication

(vi) $\{1, i, -1, -i\}$; multiplication of complex numbers