

Shortest distance from Origin to Line (9 pages; 5/6/26)

Show that the shortest distance from the line $ax + by = c$ to the Origin is $\frac{c}{\sqrt{a^2+b^2}}$, for the case where the line has a positive gradient, and a positive y -intercept.

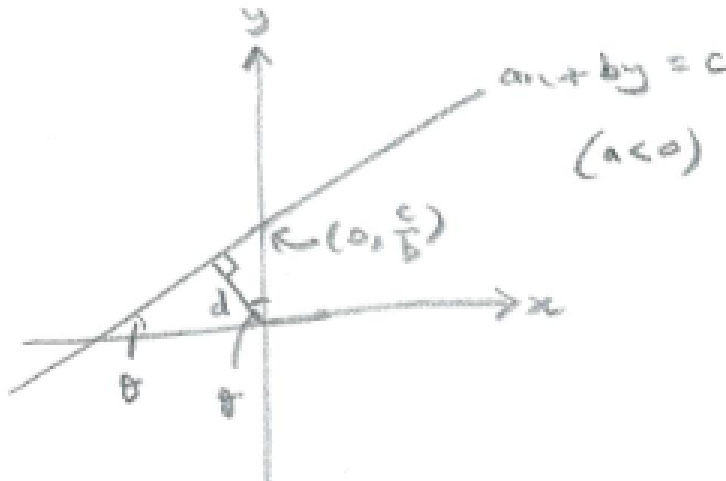
[This is analogous to the shortest distance from the plane

$n_1x + n_2y + n_3z = d$ to the Origin; namely $\frac{d}{\sqrt{n_1^2+n_2^2+n_3^2}}$]

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Solution

Method 1 [using $\tan\theta$]



Referring to the diagram, $ax + by = c \Rightarrow y = \frac{c}{b} - \frac{a}{b}x$,

so that $\tan\theta = -\frac{a}{b}$

$$\text{and } d = \frac{c}{b} \cos\theta = \frac{c}{b} \cdot \frac{1}{\sqrt{\tan^2\theta + 1}} = \frac{c}{b} \cdot \frac{1}{\sqrt{\frac{a^2}{b^2} + 1}} = \frac{c}{\sqrt{a^2 + b^2}}$$

Method 2 [vectors: shortest distance from a point to a line]

The line has vector eq'n $\underline{r} = \begin{pmatrix} 0 \\ \frac{c}{b} \end{pmatrix} + \lambda \begin{pmatrix} b \\ -a \end{pmatrix}$ (as the gradient is $-\frac{a}{b}$)

Let the point closest to the Origin (P , say) have parameter $\lambda = \lambda_P$

$$\text{Then } \overrightarrow{OP} \cdot \begin{pmatrix} b \\ -a \end{pmatrix} = 0$$

$$\text{so that } \begin{pmatrix} \lambda_P b \\ \frac{c}{b} - \lambda_P a \end{pmatrix} \cdot \begin{pmatrix} b \\ -a \end{pmatrix} = 0,$$

$$\text{and } b^2 \lambda_P - \frac{ac}{b} + a^2 \lambda_P = 0,$$

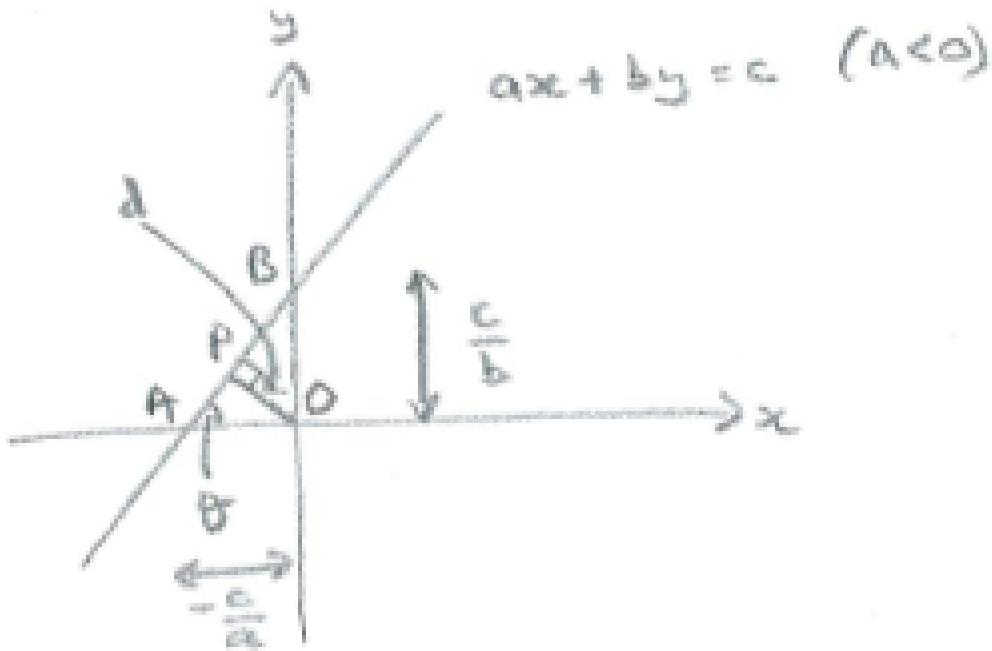
$$\text{giving } \lambda_P = \frac{ac}{b(a^2 + b^2)}$$

$$\text{Then } \overrightarrow{OP} = \frac{1}{(a^2 + b^2)} \begin{pmatrix} ac \\ \frac{c}{b}(a^2 + b^2) - \frac{a^2 c}{b} \end{pmatrix}$$

$$= \frac{1}{(a^2 + b^2)} \begin{pmatrix} ac \\ bc \end{pmatrix}$$

$$\text{and } |\overrightarrow{OP}| = \frac{c}{(a^2 + b^2)} \left| \begin{pmatrix} a \\ b \end{pmatrix} \right| = \frac{c}{(a^2 + b^2)} \sqrt{a^2 + b^2} = \frac{c}{\sqrt{a^2 + b^2}}$$

Method 3 (similar triangles)



OPA and AOB are similar triangles

$$\text{So } \frac{-(\frac{c}{a})}{d} = \frac{AB}{(\frac{c}{b})}, \text{ and } AB^2 = \left(\frac{c}{a}\right)^2 + \left(\frac{c}{b}\right)^2$$

$$\Rightarrow d = \frac{-(\frac{c}{a})(\frac{c}{b})}{AB} = \frac{-c^2}{ab\sqrt{\left(\frac{c}{a}\right)^2 + \left(\frac{c}{b}\right)^2}} = \frac{c}{\sqrt{b^2 + a^2}}$$

(noting that, as $a < 0$, $a = -\sqrt{a^2}$)

Method 4 (Intersection of perpendicular lines)

Let L be the line $ax + by = c$, and L' the line perpendicular to L .

The gradient of L is $-\frac{a}{b}$, and so the gradient of L' is $\frac{b}{a}$.

Hence the eq'n of L' is $y = \frac{b}{a}x$

Let the intersection of L & L' be (x_P, y_P) .

$$\text{Then } ax_P + b\left(\frac{b}{a}x_P\right) = c$$

$$\Rightarrow x_P(a^2 + b^2) = ac$$

$$\text{Then } d^2 = x_P^2 + y_P^2 = x_P^2\left(1 + \left(\frac{b}{a}\right)^2\right),$$

$$\text{so that } d = (-x_P)\sqrt{\frac{a^2+b^2}{a^2}} \quad (a < 0, c > 0 \Rightarrow x_P < 0)$$

$$= -\frac{ac}{a^2+b^2}\sqrt{\frac{a^2+b^2}{a^2}}$$

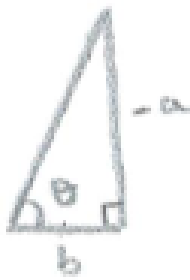
$$= \frac{c}{\sqrt{a^2+b^2}} \quad (a < 0, \text{ so that } a = -\sqrt{a^2})$$

Method 5 (trigonometry)

Referring to the diagram for Method 3,

$$\tan\theta = \frac{\left(\frac{c}{b}\right)}{\left(\frac{-c}{a}\right)} = -\frac{a}{b} \quad \text{and} \quad \sin\theta = \frac{d}{\left(\frac{-c}{a}\right)} = -\frac{ad}{c}$$

Creating a right-angled triangle where $\tan\theta = -\frac{a}{b}$, as below, shows that we can also write $\sin\theta = \frac{-a}{\sqrt{a^2+b^2}}$



Equating the two expressions for $\sin\theta$ then gives

$$-\frac{ad}{c} = \frac{-a}{\sqrt{a^2+b^2}},$$

$$\text{so that } d = \frac{c}{\sqrt{a^2+b^2}}$$

Method 6 (stationary point)

Consider a general point Q on the line $ax + by = c$ with coordinates (x, y)

Then d will be the minimum value of OQ as x varies,

$$\text{and } OQ^2 = x^2 + \left(\frac{c-ax}{b}\right)^2 \quad (1)$$

This occurs when $\frac{d}{dx}(OQ^2) = 0$;

$$\text{ie when } 2x + 2\left(\frac{c-ax}{b}\right)\left(-\frac{a}{b}\right) = 0$$

$$\Rightarrow xb^2 - ca + a^2x = 0$$

$$\Rightarrow x = \frac{ca}{a^2+b^2}$$

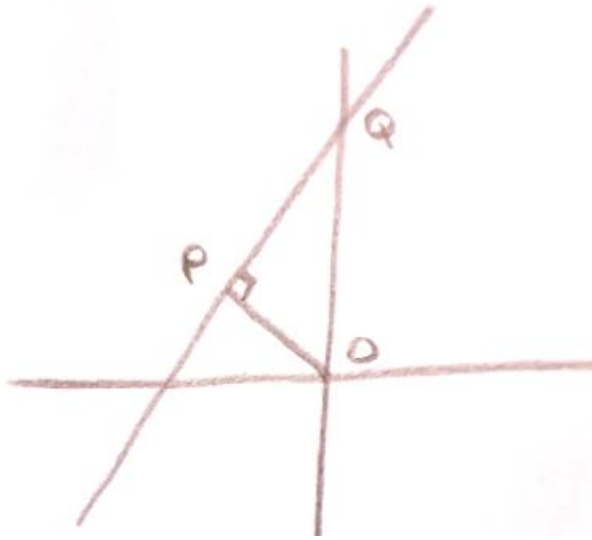
$$\text{Then, from (1), } OQ^2 = x^2 + \left(\frac{c-ax}{b}\right)^2$$

$$= \left(\frac{ca}{a^2+b^2}\right)^2 + \left(\frac{c(a^2+b^2)-ca^2}{b(a^2+b^2)}\right)^2$$

$$= \left(\frac{ca}{a^2+b^2}\right)^2 + \left(\frac{cb}{a^2+b^2}\right)^2$$

$$= \frac{c^2(a^2+b^2)}{(a^2+b^2)^2}$$

$$\text{and } d = OQ = \frac{c}{\sqrt{a^2+b^2}}$$

Method 7 (vectors: 'zigzag' method)

Write $\overrightarrow{OP} = d\underline{\hat{u}}$ and $\overrightarrow{PQ} = d'\underline{\hat{v}}$, where $\underline{\hat{u}}$ & $\underline{\hat{v}}$ are unit vectors.

[This method works where there are perpendicular lines, and no more than two unknown distances. A vector equation in 2 dimensions can be written as a pair of simultaneous equations, enabling the unknown distances to be determined.]

The gradient of the line is $-\frac{a}{b}$, and so the direction of $\underline{\hat{v}}$ is $\begin{pmatrix} b \\ -a \end{pmatrix}$

(given that $a < 0$, so that the direction of $\underline{\hat{v}}$ is correct) and

therefore $\underline{\hat{v}} = \frac{1}{M} \begin{pmatrix} b \\ -a \end{pmatrix}$, where $M = \sqrt{a^2 + b^2}$.

As $\underline{\hat{u}}$ & $\underline{\hat{v}}$ are perpendicular, $\underline{\hat{u}} = \frac{1}{M} \begin{pmatrix} a \\ b \end{pmatrix}$, so that $\underline{\hat{u}} \cdot \underline{\hat{v}} = 0$

(rather than $\frac{1}{M} \begin{pmatrix} -a \\ -b \end{pmatrix}$, so that the direction of $\underline{\hat{u}}$ is correct).

Then, as $\overrightarrow{OP} + \overrightarrow{PQ} = \overrightarrow{OQ}$, $d\underline{\hat{u}} + d'\underline{\hat{v}} = \frac{c}{b}\underline{j}$

$$\text{or } \frac{d}{M} \begin{pmatrix} a \\ b \end{pmatrix} + \frac{d'}{M} \begin{pmatrix} b \\ -a \end{pmatrix} = \frac{c}{b} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\text{Hence } \frac{d}{M} a + \frac{d'}{M} b = 0 \text{ and } \frac{db}{M} - \frac{d'}{M} a = \frac{c}{b},$$

$$\text{so that } \frac{d'}{M} = -\frac{da}{bM} \text{ and hence } \frac{db}{M} + \frac{da^2}{bM} = \frac{c}{b},$$

$$\text{giving } d(b^2 + a^2) = cM, \text{ so that } d = \frac{c}{\sqrt{a^2 + b^2}}$$