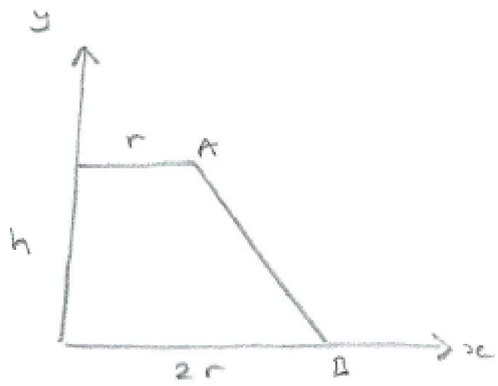
Geometry - Exercises (Sol'ns)(5 pages; 19/2/20)

(1**) Find as many ways as possible of deriving the equation of the sloping side of the trapezium shown below.



Solution

Method 1

Coordinates of A and B are (r, h) & $(2r, 0)$, so equation is:

$\frac{y-0}{x-2r} = \frac{h-0}{r-2r}$, giving $y = \frac{h(x-2r)}{-r} = 2h - \frac{hx}{r}$

Method 2

y-intercept will be $(0,2h)$ and gradient is $-\frac{h}{r}$, so equation is:

$y = 2h - \frac{hx}{r}$

Method 3a

By linear interpolation, $x = 2r - r\left(\frac{y}{h}\right)$, giving $\frac{ry}{h} = 2r - x$

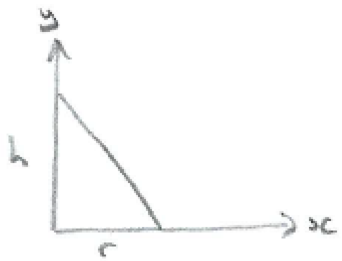
and $y = 2h - \frac{hx}{r}$

Method 3b

By linear interpolation, $y = h - h\left(\frac{x-r}{r}\right) = 2h - \frac{hx}{r}$

Method 4

The equation of the line shown below is $y = h - \frac{hx}{r}$



The required line is a translation of this line by r units to the right, and so has equation:

$y = h - \frac{h(x-r)}{r} = 2h - \frac{hx}{r}$

(2***) Find the equation of the circle passing through the points

A (2,8) , B (7,3) and D (5,7)

Solution

The first step is to find the centre of the circle, using the fact that the perpendicular bisector of each chord passes through the centre.

The chord AB has mid-point (9/2, 11/2)

and gradient $\frac{3-8}{7-2} = -1$

The perpendicular bisector of AB therefore has equation

$\frac{y-11/2}{x-9/2} = -\frac{1}{-1}$

$\rightarrow 2y - 11 = 2x - 9$

$\rightarrow y = x + 1$

The chord BD has mid-point (6, 5)

and gradient $\frac{7-3}{5-7} = -2$

The perpendicular bisector of BD therefore has equation

$\frac{y-5}{x-6} = -\frac{1}{-2}$

$\rightarrow y = \frac{1}{2}x + 2$

The centre of the circle C is then found from the intersection of these lines:

$x + 1 = \frac{1}{2}x + 2$

so that $x = 2$ and $y = 3$

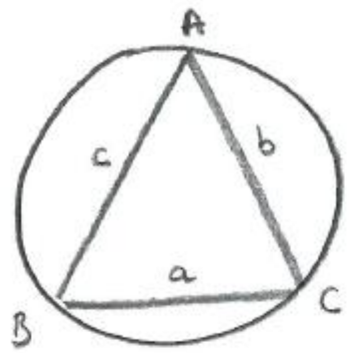
The radius is then the distance CA (for example)

$= \sqrt{(2-2)^2 + (3-8)^2} = 5$

Hence the equation of the circle is $(x-2)^2 + (y-3)^2 = 25$

(Check: B and D satisfy the equation.)

(3***) ABC is a triangle circumscribed by a circle of radius R, as shown in the diagram below.

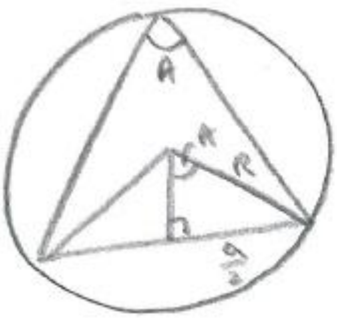


Show that (i) $\frac{a}{sinA} = 2R$ (ii) the area of the triangle is $\frac{abc}{4R}$

Solution

(i) Drawing radii from B and C to the centre of the circle, as in the diagram below, and noting that the angle at the centre is twice the angle at the circumference,

$\sin A = \frac{\left(\frac{a}{2}\right)}{R}$, so that $\frac{a}{sinA} = 2R$, as required



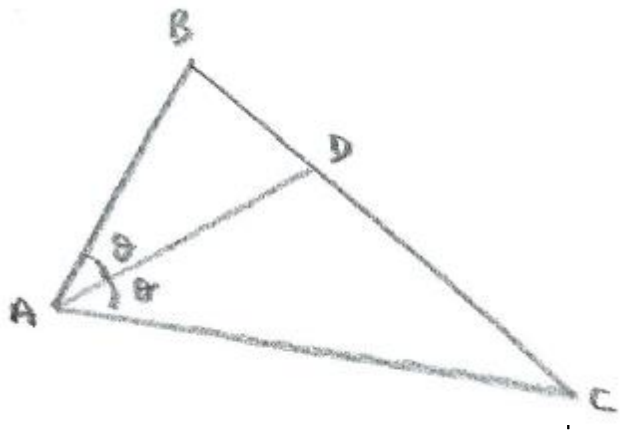
(ii) Area of $ABC = \frac{1}{2}bcsinA = \frac{1}{2}bc\left(\frac{a}{2R}\right) = \frac{abc}{4R}$

(4***) Angle Bisector Theorem

Referring to the diagram below, the Angle Bisector theorem says that

$\frac{BD}{DC} = \frac{AB}{AC}$

Prove the Angle Bisector Theorem.



Solution

By the Sine rule for triangle ABD, $\frac{BD}{sin\theta} = \frac{AB}{sinADB}$ (1)

and, for triangle ADC, $\frac{DC}{sin\theta} = \frac{AC}{sinADC} = \frac{AC}{sinADB}$ (2)

Then (1) $\Rightarrow \frac{sin\theta}{sinADB} = \frac{BD}{AB}$ and (2) $\Rightarrow \frac{sin\theta}{sinADB} = \frac{DC}{AC}$

so that $\frac{BD}{AB} = \frac{DC}{AC}$

and hence $\frac{BD}{DC} = \frac{AB}{AC}$