Geometric Distribution (3 pages; 12/11/18)

(1) Let X be the number of attempts needed to achieve a success, where the probability of success at each attempt is p (a constant).

(The attempts are assumed to be independent.)

Then $X \sim Geo(p)$ (a discrete distribution)

and $P(X = r) = (1-p)^{r-1}p$, for $r = 1,2, \ldots$

[ie $r-1$ failures, followed by a success]

(2) To confirm that $(1-p)^{r-1}p$ is a pdf:

$\sum_{r=1}^{\infty}(1-p)^{r-1}p = \frac{p}{1-(1-p)} = 1$

[being the sum of a geometric series, with 1st term p and common ratio $1-p$]

(3) $E(X) = \sum_{r=1}^{\infty} r(1-p)^{r-1}p = -p\frac{d}{dp}\sum_{r=1}^{\infty}(1-p)^r$

$= -p\frac{d}{dp}\left\{\frac{1-p}{1-(1-p)}\right\} = -p\frac{d}{dp}\left\{\frac{1}{p} - 1\right\} = -p\left(-\frac{1}{p^2}\right) = \frac{1}{p}$

(4) $E(X(X-1)) = \sum_{r=1}^{\infty} r(r-1)(1-p)^{r-1}p$

Now $\frac{d}{dp}\{(1-p)^r\} = -r(1-p)^{r-1}$

and $\frac{d}{dp}\{-r(1-p)^{r-1}\} = r(r-1)(1-p)^{r-2}$,

so that $r(r-1)(1-p)^{r-1}p = p(1-p)\{r(r-1)(1-p)^{r-2}\}$

$= p(1-p)\frac{d^2}{dp^2}\{(1-p)^r\}$

and $E(X(X-1)) = p(1-p)\sum_{r=1}^{\infty}\frac{d^2}{dp^2}\{(1-p)^r\}$

$= p(1-p)\frac{d^2}{dp^2}\{\sum_{r=1}^{\infty}(1-p)^r\}$

$= p(1-p)\frac{d^2}{dp^2}\left\{\frac{1-p}{1-(1-p)}\right\} = p(1-p)\frac{d^2}{dp^2}\{\frac{1}{p}-1\}$

$= p(1-p)\frac{d}{dp}\left\{-\frac{1}{p^2}\right\} = p(1-p)\left(\frac{2}{p^3}\right) = \frac{2(1-p)}{p^2}$

Then $Var(X) = E(X^2) - [E(X)]^2$

$= E(X(X-1)) + E(X) - [E(X)]^2$

$= \frac{2(1-p)}{p^2} + \frac{1}{p} - \frac{1}{p^2} = \frac{1}{p^2}(2-2p+p-1) = \frac{1-p}{p^2}$

(5) Cumulative probabilities

Approach A

$P(X \leq k) = 1 - P(X > k) = 1 - P(k \text{ failures})$

$= 1-(1-p)^k$

Approach B

$P(X \leq k) = \sum_{r=1}^{k}(1-p)^{r-1}p = \frac{p\{1-(1-p)^k\}}{1-(1-p)}$

[the sum of k terms of a geometric series with 1st term p and common ratio $1-p$]

$= 1-(1-p)^k$

(6) Probability generating function [see "PGFs"]

$G_X(s) = E(s^X) = \sum_{k=0}^{\infty} p_k s^k = \sum_{k=1}^{\infty} q^{k-1} p s^k$

$= ps \sum_{k=1}^{\infty} (qs)^{k-1} = \frac{ps}{1-qs}$ if $|qs| < 1$; ie $|s| < \frac{1}{q}$

$G'_X(s) = \frac{(1-qs)p - ps(-q)}{(1-qs)^2} = \frac{p}{(1-qs)^2}$

and $G''_X(s) = \frac{-2(-q)p}{(1-qs)^3} = \frac{2qp}{(1-qs)^3}$

Then $E[X] = G'_X(1) = \frac{1}{p}$

and $Var(X) = G''_X(1) + G'_X(1) - [G'_X(1)]^2$

$= \frac{2q}{p^2} + \frac{1}{p} - \frac{1}{p^2} = \frac{2(1-p)+p-1}{p^2} = \frac{1-p}{p^2}$