Game Theory - Exercises (Sol'ns) (7 pages; 16/8/19)

(1)(i) The following pay-off matrix is for a zero-sum game (from player 1's point of view).

Player 2: / Player 1	A	B	C	D
A	4	3	2	0
B	3	3	-1	-2
C	-2	2	3	1

Use the idea of dominance to reduce the matrix as much as possible.

(ii) Identify the play-safe strategies for players 1 and 2. Explain whether or not there is a stable solution.

(iii) What will be the outcome of the game if:
(a) both players play safe
(b) player 1 plays safe, and player 2 hears of player 1's intention
(c) player 2 plays safe, and player 1 hears of player 2's intention

Solution

(i) Row A dominates row B, and column D dominates columns B and C (as player 2 will always prefer D to B or C). The reduced matrix is:

Player 2: / Player 1	A	D
A	4	0
C	-2	1

(ii)

Player 2: Player 1	A	D	row min.
A	4	0	(0)
C	-2	1	-2
col. max.	4	(1)	

The play-safe strategy for player 1 is A, and for player 2 it is D.

As the min. of the col. maxima does not equal the max. of the row minima, there is no stable solution.

(iii)(a) neither player wins anything

(b) player 1 chooses A and player 2 then chooses D, so that neither player wins anything

(c) player 2 chooses D and player 1 then chooses C, so that player 1 wins 1 and player 2 loses 1

(2) A zero-sum game is given by the following pay-off matrix (from player 1's point of view). Confirm that there is no stable solution, and find the optimal mixed strategy for each player, and their expected pay-offs.

Player 2: Player 1	A	B
A	2	3
B	4	-1

Solution

Player 2: Player 1	A	B	row min.
A	2	3	(2)
B	4	-1	-1
col. max.	4	(3)	

As the max. of the row minima doesn't equal the min. of the col. maxima, there is no stable solution.

Let p be the probability that player 1 chooses option A.

Then the expected pay-off for player 1 if player 2 chooses A is:

$2p + 4(1-p) = 4 - 2p$

and if player 2 chooses B it is:

$3p + (-1)(1-p) = 4p - 1$

The optimal value of p occurs when $min(4-2p, 4p-1)$ is maximised, and this occurs at the intersection of the lines $y = 4 - 2p$ and $y = 4p - 1$

ie when $4 - 2p = 4p - 1 \Rightarrow 6p = 5; p = \frac{5}{6}$

and the expected pay-off for player 1 is $4 - 2\left(\frac{5}{6}\right) = \frac{7}{3}$

Similarly, let q be the probability that player 2 chooses option A.

Then the expected pay-off for player 2 if player 1 chooses A is:

$(-2)q + (-3)(1-q) = q - 3$

and if player 1 chooses B it is:

$(-4)q + (1)(1-q) = 1 - 5q$

The optimal value of q occurs when $min(q-3, 1-5q)$ is maximised, and this occurs at the intersection of the lines $y = q - 3$ and $y = 1 - 5q$

ie when $q - 3 = 1 - 5q \Rightarrow 6q = 4; q = \frac{2}{3}$

and the expected pay-off for player 2 is $\frac{2}{3} - 3 = -\frac{7}{3}$

[$-1 \times$ player 1's expected pay-off, as would be expected]

(3) A zero-sum game is given by the following pay-off matrix (from player 1's point of view).

Player 2: / Player 1	A	B	C
A	1	-2	2
B	3	4	-1

(i) Confirm that there is no stable solution, and find the optimal mixed strategy for player 1, and their expected pay-off.

(ii) By using the fact that the expected pay-off for player 2 will equal $-1 \times$

the expected pay-off for player 1, find the optimal mixed strategy for player 2.

Solution

(i)

Player 2: Player 1	A	B	C	row min.
A	1	-2	2	-2
B	3	4	-1	(-1)
col. max.	3	4	(2)	

As the max. of the row minima doesn't equal the min. of the col. maxima, there is no stable solution.

To find the strategy for player 1:

Let p be the probability that player 1 chooses option A.

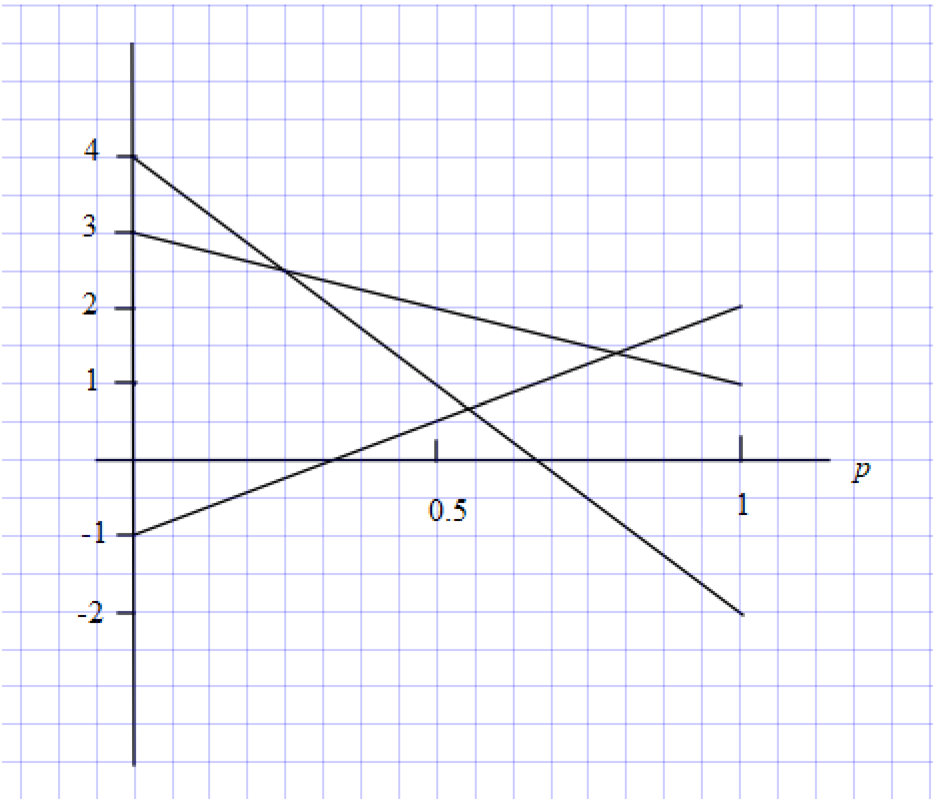
Then the expected pay-off for player 1 if player 2 chooses A is:

$p + 3(1 - p) = 3 - 2p$

If player 2 chooses B it is: $-2p + 4(1 - p) = 4 - 6p$

And if player 2 chooses C it is: $2p - (1 - p) = 3p - 1$

The diagram below shows the lines $y = 3 - 2p$, $y = 4 - 6p$ and $y = 3p - 1$



The optimal value of p occurs when $min(3-2p\,,4-6p,3p-1)$ is maximised, and this occurs at the intersection of the lines

$y=4-6p$ and $y=3p-1$

ie when $4-6p=3p-1\Rightarrow 5=9p;p=\frac{5}{9}$

and the expected pay-off for player 1 is $4-6\left(\frac{5}{9}\right)=\frac{2}{3}$

To find the strategy for player 2:

Let q be the probability that player 2 chooses option A, and r the probability that they choose option B; so that they choose option C with probability

$1-q-r$.

Then the expected pay-off for player 2 if player 1 chooses A is:

$(-1)q+2r-2(1-q-r)=q-2+4r$

If player 1 chooses B it is: $-3q - 4r + (1 - q - r) = -4q - 5r + 1$

The probability rule is chosen in such a way that the expected pay-offs are the same , whichever option player 1 chooses, and both are equal to the value of the game from player 2's point of view, which is known to be $-\frac{2}{3}$.

So $q + 4r - 2 = -\frac{2}{3}$ and $-4q - 5r + 1 = -\frac{2}{3}$

$\Rightarrow 3q + 12r - 6 = -2$ or $3q + 12r = 4$ (1)

and $-12q - 15r + 3 = -2$ or $12q + 15r = 5$ (2)

Then $4 \times (1) - (2) \Rightarrow 33r = 11; r = \frac{1}{3} \Rightarrow q = 0$

So the probability rule for player 2 is:

Choose A with probability 0

Choose B with probability $\frac{1}{3}$

Choose C with probability $\frac{2}{3}$